Problem I [**Ginsparg-Wilson relation**]

Suppose one defines a (free) lattice Dirac fermion from the continuum theory through the following block-spin transformation:

$$e^{-\sum_{mn}\bar\psi_m h_{mn}\psi_n} = \int D[\psi]D[\bar\psi]\, e^{-\int d^4x\,\bar\psi(x)\gamma_\mu\partial_\mu\psi(x)} \times e^{-\sum_{m,n}\left(\bar\psi_m-\int_{V_m}d^4x\bar\psi(x)\right)\alpha_{mn}\left(\psi_n-\int_{V_n}d^4x\psi(x)\right)} \tag{1}$$

$V_n = a^4$ is a hypercubic volume element located at a lattice point $x = na$. α_{mn} commutes with γ_5. Show that the lattice Dirac op. h_{mn} satisfies the Ginsparg-Wilson relation

$$\gamma_5 h + h\gamma_5 = 2h\gamma_5\alpha^{-1}h. \tag{2}$$

If one applies chiral transformation to the lattice Dirac field as

$$\delta\psi_n = i\epsilon\gamma_5\psi_n, \delta\bar\psi_n = i\epsilon\bar\psi_n\gamma_5$$

the L.H.S. reads

$$\delta\,(\text{L.H.S.}) = -i\epsilon\sum_{mn}\bar\psi_m\{\gamma_5 h_{mn} + h_{mn}\gamma_5\}\psi_n\, e^{-\sum_{mn}\bar\psi_m h_{mn}\psi_n}$$

On the other hand, in the R.H.S. one may also apply chiral transformation to the continuum Dirac fied at the same time as

$$\delta\psi(x) = i\epsilon\gamma_5\psi(x), \delta\bar\psi(x) = i\epsilon\bar\psi(x)\gamma_5$$

Since the continuum action is invariant under the chiral transformation, the R.H.S. reads

$$\begin{aligned}\delta\,(\text{R.H.S.}) &= \int D[\psi]D[\bar\psi]\, e^{-\int d^4x\,\bar\psi(x)\gamma_\mu\partial_\mu\psi(x)} \times \\ &\quad e^{-\sum_{m,n}\left(\bar\psi_m-\int_{V_m}d^4x\bar\psi(x)\right)\alpha_{mn}\left(\psi_n-\int_{V_n}d^4x\psi(x)\right)} \times \\ &\quad (-i\epsilon)\sum_{m,n}\left(\bar\psi_m - \int_{V_m}d^4x\bar\psi(x)\right)2\gamma_5\alpha_{mn}\left(\psi_n - \int_{V_n}d^4x\psi(x)\right)\end{aligned}$$

(Here we assumes that the fermion measure $D[\psi]D[\bar\psi]$ in the continuum theory is invariant under the chiral transformation.) The above result may be rewritten using the derivative w.r.t. to the lattice Dirac field (treating it as an external source) :

$$\begin{aligned}\delta\,(\text{R.H.S.}) &= \int D[\psi]D[\bar\psi]\,(-i\epsilon)\sum_{mn}\left(\frac{\delta}{\delta\psi}\alpha^{-1}\right)_m 2\gamma_5\alpha_{mn}\left(\alpha^{-1}\frac{\delta}{\delta\bar\psi}\right)_n \\ &\quad e^{-\int d^4x\,\bar\psi(x)\gamma_\mu\partial_\mu\psi(x)} \times e^{-\sum_{m,n}\left(\bar\psi_m-\int_{V_m}d^4x\bar\psi(x)\right)\alpha_{mn}\left(\psi_n-\int_{V_n}d^4x\psi(x)\right)} \\ &= (-i\epsilon)\sum_{mn}\left(\frac{\delta}{\delta\psi}\alpha^{-1}\right)_m 2\gamma_5\alpha_{mn}\left(\alpha^{-1}\frac{\delta}{\delta\bar\psi}\right)_n e^{-\sum_{mn}\bar\psi_m h_{mn}\psi_n} \\ &= -i\epsilon\sum_{mn}\bar\psi_m\{h\alpha^{-1}(2\gamma_5\alpha)\alpha^{-1}h\}_{mn}\psi_n\, e^{-\sum_{mn}\bar\psi_m h_{mn}\psi_n}\end{aligned}$$

(Here we also assumes $\mathrm{Tr}\gamma_5\alpha = 0$, $\mathrm{Tr}\gamma_5\alpha^{-1}h = 0$) Then the Ginsparg-Wilson relation (2) follows.

Problem II [**Free Overlap Fermion**]

Set $U(x,\mu)=1$, $a=1$ and derive the Fourier transformation of the free overlap Dirac operator, $\tilde{D}(p)$,

$$D\,\mathrm{e}^{ipx} = \tilde{D}(p)\,\mathrm{e}^{ipx}$$

Show the following four properties of $\tilde{D}(p)$,

1. $\tilde{D}(p)$ is a periodic and analytic function of p_μ
2. $\tilde{D}(p) = Zi\gamma_\mu p_\mu + O(p^2)$ $(p \ll \pi)$
3. $\tilde{D}(p) \simeq 1$ $(|p_\mu| \simeq \pi)$
4. $\gamma_5 S(p) + S(p)\gamma_5 = 2\gamma_5,\ \ S(p) = \tilde{D}(p)^{-1}$

The forward- and backward-difference operators act on the plane wave as follows:

$$\partial_\mu\,\mathrm{e}^{ipx} = (\mathrm{e}^{ip_\mu} - 1)\,\mathrm{e}^{ipx}, \quad \partial_\mu^*\,\mathrm{e}^{ipx} = (1 - \mathrm{e}^{-ip_\mu})\,\mathrm{e}^{ipx}.$$

$$\bar{p}_\mu = \sin p_\mu, \quad \hat{p}_\mu = 2\sin\frac{p_\mu}{2} \tag{3}$$

The F.T. of the Wilson-Dirac operator then reads

$$(D_w - m_0)\,\mathrm{e}^{ipx} = \left(i\gamma_\mu\bar{p}_\mu + \frac{1}{2}\hat{p}^2 - m_0\right)\mathrm{e}^{ipx}.$$

Similary, the F.T. of the overlap Dirac operator is obtained as follows:

$$\begin{aligned} D\,\mathrm{e}^{ipx} &= \frac{1}{2}\left(1 + (D_w - m_0)\frac{1}{\sqrt{(D_w - m_0)^\dagger (D_w - m_0)}}\right)\mathrm{e}^{ipx} \\ &= \frac{1}{2}\left(1 + \frac{i\gamma_\mu\bar{p}_\mu + \frac{1}{2}\hat{p}^2 - m_0}{\sqrt{\bar{p}^2 + \left(\frac{1}{2}\hat{p}^2 - m_0\right)^2}}\right)\mathrm{e}^{ipx} \end{aligned}$$

$$\therefore\ \tilde{D}(p) = \frac{1}{2}\left(1 + \frac{i\gamma_\mu\bar{p}_\mu + \frac{1}{2}\hat{p}^2 - m_0}{\sqrt{\bar{p}^2 + \left(\frac{1}{2}\hat{p}^2 - m_0\right)^2}}\right)$$

1. Periodicity is obvious from Eqs. (3). Analyticity could be broken down if and only if the inside of the square root in the denominater vanishes identically. Namely,

   $$\bar{p}^2 + \left(\frac{1}{2}\hat{p}^2 - m_0\right)^2 = 0$$

   or

   $$\bar{p} = 0 \quad \text{and} \quad \frac{1}{2}\hat{p}^2 - m_0 = 0$$

   But the first condition implies $\hat{p}^2 = 4n (n : \text{integer})$ and therefore the second condition is never fulfilled for $0 < m_0 < 2$.

2.

$$\tilde{D}(p) \simeq \frac{1}{2m_0} i\gamma_\mu p_\mu \qquad (|p_\mu| \ll 1)$$

3.

$$\tilde{D}(p) \simeq \frac{1}{2}\left(1 + \frac{2n - m_0}{\sqrt{(2n-m_0)^2}}\right) = 1$$

$|p_\mu| \simeq \pi$, or 0. $n(\neq 0)$ is the number of the components of p_μ which is equal to π.

4. If one set $\omega(p) = \sqrt{\bar{p}^2 + \left(\frac{1}{2}\hat{p}^2 - m_0\right)^2}$, then $\tilde{D}(p)$ reads

$$\tilde{D}(p) = \frac{i\gamma_\mu \bar{p}_\mu + \omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]}{2\omega(p)}$$

Then the free fermion propagator is evaluated as follows:

$$S(p) = 2\omega(p)\frac{-i\gamma_\mu \bar{p}_\mu + \omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]}{\bar{p}^2 + \left(\omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]\right)^2}$$

where

$$\begin{aligned}
\bar{p}^2 + \left(\omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]\right)^2 &= \bar{p}^2 + \left[\frac{1}{2}\hat{p}^2 - m_0\right]^2 + \omega(p)^2 + 2\omega(p)\left[\frac{1}{2}\hat{p}^2 - m_0\right] \\
&= 2\omega(p)\left(\omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]\right)
\end{aligned}$$

Therefore it follows that

$$S(p) = \frac{-i\gamma_\mu \bar{p}_\mu}{\omega(p) + \left[\frac{1}{2}\hat{p}^2 - m_0\right]} + 1, \qquad \gamma_5 S(p) + S(p)\gamma_5 = 2\gamma_5$$

Problem III [**Admissibility condition**]

For $m_0 = 1$, $H_w^2 = (aD_w - 1)^\dagger (aD_w - 1)$ is evaluated as follows:

$$(aD_w - 1)^\dagger (aD_w - 1) = 1 + \frac{1}{4} \sum_{\mu \neq \nu} \{ B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu} \}$$

$$\begin{aligned}
B_{\mu\nu} &= a^4 \nabla_\mu^* \nabla_\mu \nabla_\nu^* \nabla_\nu = a^4 \nabla_\mu^* \nabla_\nu^* \nabla_\nu \nabla_\mu - a^3 \nabla_\mu^* \left[\nabla_\mu, \nabla_\nu^* - \nabla_\nu \right] \\
C_{\mu\nu} &= \frac{1}{2} i \sigma_{\mu\nu} a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu \right] \\
D_{\mu\nu} &= -\gamma_\mu a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu \right]
\end{aligned}$$

Show

$$a^2 H_w^2 \geq 1 - 30\epsilon$$

$$\begin{aligned}
a \nabla_\mu \phi(x) &= U(x,\mu)\phi(x+\hat{\mu}) - \phi(x), \\
a^2 \nabla_\nu \nabla_\mu \phi(x) &= U(x,\nu)\left(U(x+\hat{\nu},\mu)\phi(x+\hat{\mu}+\hat{\nu}) - \phi(x+\hat{\nu}) \right) - \left(U(x,\mu)\phi(x+\hat{\mu}) - \phi(x) \right), \\
a^2 \left[\nabla_\mu, \nabla_\nu \right] \phi(x) &= \left\{ U(x,\mu)U(x+\hat{\mu},\nu) - U(x,\nu)U(x+\hat{\nu},\mu) \right\} \phi(x+\hat{\mu}+\hat{\nu}) \\
&= -\left\{ 1 - U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1} \right\} \times \\
&\qquad U(x,\nu)U(x+\hat{\nu},\mu)\phi(x+\hat{\mu}+\hat{\nu})
\end{aligned}$$

Therefore

$$\parallel 1 - U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1} \parallel \leq \epsilon \quad \Rightarrow \quad \parallel a^2 \left[\nabla_\mu, \nabla_\nu \right] \parallel \leq \epsilon$$

$$\begin{aligned}
a^2 H_w^2 &\geq 1 - \frac{1}{4} \sum_{\mu \neq \nu} \parallel -a^3 \nabla_\mu^* \left[\nabla_\mu, \nabla_\nu^* - \nabla_\nu \right] \parallel \\
&\quad - \frac{1}{4} \sum_{\mu \neq \nu} \parallel \frac{1}{2} i \sigma_{\mu\nu} a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu \right] \parallel - \frac{1}{4} \sum_{\mu \neq \nu} \parallel -\gamma_\mu a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu \right] \parallel \\
&= 1 - \frac{1}{4} \cdot 12 \cdot 2 \cdot 2\epsilon - \frac{1}{4} \cdot \frac{1}{2} \cdot 12 \cdot 4\epsilon - \frac{1}{4} \cdot 12 \cdot 4\epsilon = 1 - 30\epsilon
\end{aligned}$$

cf. (in the continuum limit)

$$\begin{aligned}
\left(a\gamma_\mu D_\mu - 1 \right)^\dagger \left(a\gamma_\mu D_\mu - 1 \right) &= 1 - a^4 (D_\mu)^2 - a^4 \frac{\left[\gamma_\mu, \gamma_\nu \right]}{4} \left[D_\mu, D_\nu \right] \\
&= 1 - a^4 (D_\mu)^2 - i a^4 \frac{\left[\gamma_\mu, \gamma_\nu \right]}{4} F_{\mu\nu}
\end{aligned}$$

Problem IV [**Locality of overlap Dirac operator**]

When $a^2H_w^2 = (aD_w - 1)^\dagger(aD_w - 1)$ satisfies the bound $0 < \alpha <\| a^2H_w^2 \|< \beta$, the inverse square root of $a^2H_w^2$ can be expanded through the Legendre polynomials as follows:

$$\frac{1}{\sqrt{a^2H_w^2}} = \frac{\kappa}{\sqrt{1-2tz+t^2}} = \kappa\sum_{k=0}^{\infty} t^k P_k(z), \quad z = \frac{\beta+\alpha-2H_w^2}{\beta-\alpha},$$

where

$$\cosh\theta = \frac{\beta-\alpha}{\beta+\alpha}, \quad t = e^{-\theta}, \quad \kappa = \sqrt{\frac{4t}{\beta-\alpha}}.$$

Prove the bound

$$\left\| \frac{1}{\sqrt{a^2H_w^2}}(x,y) \right\| < \frac{\kappa}{1-t}\ \exp\{-\theta|x-y|/2a\}.$$

($|x-y|$ is the taxi driver distance between the lattice sites x and y.)

Since Wilson-Dirac operator D_w involves only nearest-neighbor couplings, $a^2H_w^2$ and z involve at most next-to-nearest-neighbor couplings. Then for a fixed $|x-y|$, the Legendre polynomial expansion of the inverse square root of $a^2H_w^2$ and z starts from the order k such that $2k \geq |x-y|/a$:

$$\frac{1}{\sqrt{a^2H_w^2}}(x,y) = \kappa \sum_{k\geq|x-y|/2a} t^k P_k(z).$$

Since the Legendre polynomials statify the bound $\|P_k(z)\| \leq 1$, one has

$$\left\| \frac{1}{\sqrt{a^2H_w^2}}(x,y) \right\| < \kappa \sum_{k\geq|x-y|/2a} t^k \|P_k(z)\| = \sum_{k=0}^{\infty} t^k\ \exp\{-\theta|x-y|/2a\} = \frac{\kappa}{1-t}\ \exp\{-\theta|x-y|/2a\}.$$

Problem V [**Index theorem on the lattice**]

Overlap Dirac operator is normal and satisfies the γ_5-conjugate relation:

$$D + D^\dagger = 2aD^\dagger D = 2aDD^\dagger \text{ (normal)}, \quad D^\dagger = \gamma_5 D \gamma_5 \text{ (γ_5-conjugate)}$$

1. Show that the eivenvalues of D,

$$D\psi_\lambda = \lambda\psi_\lambda,$$

distribute on the circle with the radius $1/2a$ and centered at $(1/2a, 0)$ in the two-dimensional complex plane.

2. Show that the eivenvalues of D are classified into three groups as follows:

$$\begin{array}{rll} \lambda = 0: & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & n_\pm \\ \lambda = 1/a: & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & N_\pm \\ \lambda \neq 0, 1/a: & \text{pair-wise} \left\{ \begin{array}{l} \lambda \to \psi_\lambda \\ \lambda^* \to \gamma_5\psi_\lambda \end{array} \right. & \end{array}$$

3. Prove the index theorem on the lattice

$$\mathrm{Tr}\{\gamma_5(1 - aD)\} = n_+ - n_-$$

---

1. For an eigenvalue λ and an eivenvector $\psi_\lambda(x)$ belonging to it,

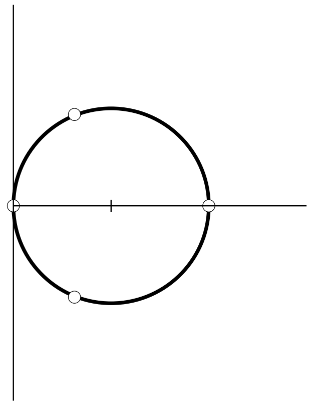
$$D\,\psi_\lambda(x) = \lambda\,\psi_\lambda(x),$$

one has

$$a^4 \sum_x \psi_\lambda^\dagger(x) \left\{ D + D^\dagger - 2aD^\dagger D \right\} \psi_\lambda(x) = (\lambda + \lambda^* - 2a\lambda^*\lambda)\,(\psi_\lambda, \psi_\lambda) = 0$$

Then one can show

$$\lambda + \lambda^* - 2a\lambda^*\lambda = (-2a)\left[(\lambda - 1/2a)(\lambda - 1/2a)^* - (1/2a)^2\right] = 0$$



2. (a) $\lambda = 0$

Suppose ψ_λ belongs to the zero eigenvalue, $D\psi_\lambda = 0$. Then

$$D(\gamma_5\psi_\lambda) = \{-\gamma_5 D + 2aD\gamma_5 D\}\psi_\lambda = 0.$$

This implies that ψ_λ can be made to be chiral

$$D\left(\frac{1+\gamma_5}{2}\right)\psi_\lambda = 0 \quad \text{or} \quad D\left(\frac{1-\gamma_5}{2}\right)\psi_\lambda = 0$$

(b) $\lambda = 1/a$

Suppose ψ_λ belongs to the eigenvalue $1/a$, $D\psi_\lambda = (1/a)\psi_\lambda$. Then

$$(D - 1/a)(\gamma_5\psi_\lambda) = \{-\gamma_5(D - 1/a) + 2a(D - 1/a)\gamma_5(D - 1/a)\}\psi_\lambda = 0.$$

This implies that ψ_λ can be made to be chiral

$$(D - 1/a)\left(\frac{1+\gamma_5}{2}\right)\psi_\lambda = 0 \quad \text{or} \quad (D - 1/a)\left(\frac{1-\gamma_5}{2}\right)\psi_\lambda = 0$$

(c) $\lambda \neq 0, 1/a$

Suppose ψ_λ belongs to the eigenvalue $\lambda \neq 0, 1/a$, $D\psi_\lambda = \lambda\psi_\lambda$. Since D is normal, $\psi_\lambda^\dagger D = \lambda\psi_\lambda^\dagger$. Then

$$D(\gamma_5\psi_\lambda) = \gamma_5 D^\dagger\psi_\lambda = \gamma_5\left(\lambda\psi_\lambda^\dagger\right)^\dagger = \lambda^*(\gamma_5\psi_\lambda)$$

3.

$$\begin{aligned}
\sum_\lambda \psi_\lambda^\dagger\gamma_5(1 - aD)\psi_\lambda &= \sum_{\lambda=0,1/a} \psi_\lambda^\dagger\gamma_5\psi_\lambda - a\sum_{\lambda=1/a} \lambda\,\psi_\lambda^\dagger\gamma_5\psi_\lambda \\
&= (n_+ - n_-) + (N_+ - N_-) - (N_+ - N_-) \\
&= n_+ - n_-
\end{aligned}$$

Problem VI [**Numerical check of topological charge on the lattice**]

Consider U(1) gauge fields on the two-dimensional lattice with the finite volume $V = La \times La$ (L: integer). Topological charge of the gauge fields may be defined through

$$Q_g = \frac{1}{2\pi} \sum_x F_{\mu\nu}(x)$$

where

$$F_{\mu\nu}(x) = \frac{1}{i} \ln \left\{ U(x,\mu) U(x+\hat{\mu},\nu) U(x+\hat{\nu},\mu)^{-1} U(x,\nu)^{-1} \right\}$$

1. For the $U(1)$ instanton configuration given by

   $$V_{[m]}(x,1) = \exp\left\{ -\frac{2\pi i}{L} m x_2 \, \delta_{x_1, L-1} \right\}, \quad V_{[m]}(x,2) = \exp\left\{ +\frac{2\pi i}{L^2} m x_1 \right\} \qquad (m: \text{ integer }),$$

   compute Q_g and show $Q_g = m$.

2. For $V_{[m]}(x,\mu)$ with $L = 8$, compute all eigenvalues of H_w numerically and evalute $Q = -\frac{1}{2}\mathrm{Tr}\left(\frac{H_w}{\sqrt{H_w}}\right)$.

1.

2. A sample fortran program to compute the eivenvalues and eivenvectors of H_w is available from

   http://www.eken.phys.nagoya-u.ac.jp/~kikukawa/LFT/

Problem VII [**Propagator of Ginsparg-Wilson Weyl fermion**]

Derive the Weyl fermion propergator

$$\langle \psi_L(x)\, \bar{\psi}_L(y) \rangle_F = \hat{P}_L D^{-1} P_R$$

By performing the grassman number integration over c_i, $\bar{c}_k$, one obtains

$$\langle \psi_L(x)\, \bar{\psi}_L(y) \rangle_F = \sum_{i,k} v_i(x) \left(M^{-1}\right)_{ik} \bar{v}_k(y), \qquad M_{ki} = (\bar{v}_k D v_i).$$

Noting

$$\hat{P}_L(x,y) = \sum_i v_i(x) v_i(y)^\dagger, \;\; P_R(x,y) = \sum_k \bar{v}_k(x)^\dagger \bar{v}_k(y),$$

one can show

$$\begin{aligned} &\sum_i (\bar{v}_k D v_i)(v_i^\dagger D^{-1} \bar{v}_l^\dagger) \\ &= \bar{v}_k D \hat{P}_L D^{-1} \bar{v}_l^\dagger = \bar{v}_k P_R D D^{-1} \bar{v}_l^\dagger = \delta_{kl}, \end{aligned}$$

which implies

$$\left(M^{-1}\right)_{ik} = \sum_x v_i(x)^\dagger D^{-1} \bar{v}_k(x)^\dagger.$$

Then one obtains

$$\begin{aligned} \langle \psi_L(x)\, \bar{\psi}_L(y) \rangle_F &= \sum_{i,k} v_i(x) \left(M^{-1}\right)_{ik} \bar{v}_k(y) \\ &= \sum_{i,k} v_i(x)\, v_i(x)^\dagger D^{-1} \bar{v}_k(x)^\dagger\, \bar{v}_k(y) \\ &= \hat{P}_L D^{-1} P_R. \end{aligned}$$

Problem VIII [**Nambu-Goldstone theorem**]

Consider lattice QCD with massive Ginsparg-Wilson fermions,

$$S = S_G + a^4 \sum_x \bar{\psi}(x)(D+m)\psi(x).$$

Then, the zero-momentum limit of the Fourier transform of the two-point function of the psuedo scalar operator (pion field) is given as follows:

$$G_{ab} = \frac{1}{\Omega} \sum_{x,y} \left\langle \bar{\psi}\tau^a \gamma_5 \psi(x)\, \bar{\psi}\tau^a \gamma_5 \psi(y) \right\rangle = \delta_{ab} \frac{1}{\Omega a^8} \left\langle \mathrm{Tr}\left(\frac{1}{D+m}\gamma_5 \frac{1}{D+m}\gamma_5 \right) \right\rangle$$

(Ωa^4 is the space-time volume)

Show followings by using the Ginsparg-Wilson relation.

1.
$$(1+2am)[(D+m)\gamma_5 + \gamma_5(D+m)] = m(2+2am)\gamma_5 + 2a(D+m)\gamma_5(D+m)$$

2.
$$\mathrm{Tr}\left(\frac{1}{D+m}\gamma_5 \frac{1}{D+m}\gamma_5 \right) = \frac{(1+2am)}{m(1+am)} \mathrm{Tr}\left(\frac{1}{D+m} - a\frac{1}{(1+2am)} \right)$$

3.
$$\lim_{m\to 0} G_{ab} = \delta_{ab} \frac{1}{ma^4} \left\langle \bar{\psi}(1-aD)\psi \right\rangle$$

Problem IX [**Domain-wall fermion**]

Domain-wall fermion is defined with the five-dimensional Wilson-Dirac fermion with the Dirichlet b.c. in the fifth direction :

$$S_{\rm DWF} = a_5 \sum_{t=1}^{N} \sum_x \bar{\Psi}(x,t)(D_{5w} - m_0)\Psi(x,t)\Big|_{\rm Dir.} \qquad (a=1,\ 0<m_0<1)$$

Show that the partition function of the domain wall fermion is factorized into two parts as follows:

$$\det\left(D_{5\rm w} - m_0\right)_{[Dir]} = \det D_N \cdot \det\left(D_{5\rm w} - m_0\right)_{[AP]}$$

where D_N is a four-dimensional Dirac operator given by

$$D_N = \frac{1}{2a}\left(1 + \gamma_5 \tanh\left(\frac{a_5 N H}{2}\right)\right)$$

and

$$H = -\frac{1}{a_5}\ln T$$

$$T = \begin{pmatrix} \frac{1}{B} & \frac{1}{B}a_5 C \\ -a_5 C^\dagger \frac{1}{B} & B + a_5^2 C^\dagger \frac{1}{B} C \end{pmatrix}$$

$$B = 1 + a_5(B_4 - m_0), \qquad D_{\rm w} = \begin{pmatrix} B_4 & -C \\ C^\dagger & -B_4 \end{pmatrix}$$

See next page

Evaluation of det $\left(D_{5\mathrm{w}} - \frac{m_0}{a}\right)$ $(a_5 = a = 1)$

$$
\begin{aligned}
& D_{5\mathrm{w}} - m_0 \\
= \; & \left(D_{\mathrm{w}} - m_0 + 1\right) \delta_{ts} - P_L \delta_{t+1,s} - P_R \delta_{t,s+1}
\end{aligned}
$$

$$
= \begin{pmatrix}
\begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} +1 & 0 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot & \cdot \\
\cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot \\
\cdot & \cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ 0 & +1 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix}
\end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix}
\begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & +1 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot & \cdot \\
\cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot \\
\cdot & \cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ +1 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix}
\end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix}
\begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot & \cdot & \begin{bmatrix} +1 & C \\ 0 & B \end{bmatrix} \\
\begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot & \cdot \\
\cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot \\
\cdot & \cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot \\
\cdot & \cdot & \cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & +1 \end{pmatrix}
\end{pmatrix}
$$

Set

$$\alpha \equiv \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} \quad \beta \equiv \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix}$$

$$\alpha_X \equiv \begin{pmatrix} B & 0 \\ -C^\dagger & X \end{pmatrix} \quad \beta_Y \equiv \begin{bmatrix} Y & C \\ 0 & B \end{bmatrix}$$

$$X, Y = 0 \text{ for Dirichlet B.C., } X, Y = +1 \text{ for AP B.C.}$$

Then

$$D_{5\mathrm{w}} - m_0 \;\; \Rightarrow \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & \beta_Y \\ \beta & \alpha & \cdot & \cdot & \cdot \\ \cdot & \beta & \alpha & \cdot & \cdot \\ \cdot & \cdot & \beta & \alpha & \cdot \\ \cdot & \cdot & \cdot & \beta & \alpha_X \end{pmatrix} = \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & \cdot \\ \beta & \alpha & \cdot & \cdot & \cdot \\ \cdot & \beta & \alpha & \cdot & \cdot \\ \cdot & \cdot & \beta & \alpha & \cdot \\ \cdot & \cdot & \cdot & \beta & \alpha_X \end{pmatrix} \times \begin{pmatrix} 1 & \cdot & \cdot & \cdot & -V_1 \\ \cdot & 1 & \cdot & \cdot & -V_2 \\ \cdot & \cdot & 1 & \cdot & -V_3 \\ \cdot & \cdot & \cdot & 1 & -V_4 \\ \cdot & \cdot & \cdot & \cdot & 1 - V_5 \end{pmatrix}$$

where

$$\begin{aligned} -\alpha V_1 &= \beta_Y \\ -\beta V_1 - \alpha V_2 &= 0 \\ -\beta V_2 - \alpha V_3 &= 0 \\ -\beta V_3 - \alpha V_4 &= 0 \\ -\beta V_4 + \alpha_X (1 - V_5) &= \alpha_X \end{aligned}$$

Now one can evaluate the determinant:

$$\det \left(D_{5\mathrm{w}} - m_0 \right)_{X,Y} = \{\det \alpha\}^{N-1} \det \alpha_X \det (1 - V_N)$$

where

$$V_N = \alpha_X^{-1} \alpha \cdot \left\{ -\alpha^{-1} \beta \right\}^N \cdot \beta^{-1} \beta_Y$$

$$\begin{aligned} -\alpha^{-1}\beta &= \begin{pmatrix} \frac{1}{B} & -\frac{1}{B} C \\ -C^\dagger \frac{1}{B} & B + C^\dagger \frac{1}{B} C \end{pmatrix} = T = e^{-H} \\ \alpha^{-1}\alpha_Y &= \begin{pmatrix} 1 & 0 \\ 0 & -X \end{pmatrix} \\ \beta^{-1}\beta_Y &= \begin{pmatrix} -Y & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

For Dirichlet b.c. and Anti-Periodic b.c., respectively, one obtains

$$\det \left(D_{5\mathrm{w}} - m_0 \right) = \{\det (P_L + P_R B)\}^N \cdot \det \gamma_5 \cdot \det (P_R + P_L T^N)$$

$$\det \left(D_{5\mathrm{w}} - m_0 \right)_{AP} = \{\det (P_L + P_R B)\}^N \cdot \det \gamma_5 \cdot \det (1 + T^N)$$

$$\frac{P_R + P_L T^N}{1 + T^N} = \frac{1}{2} \left(1 + \gamma_5 \frac{1 - T^N}{1 + T^N} \right) = \frac{1}{2} \left(1 + \gamma_5 \tanh (NH/2) \right) = D_N$$

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