まとめ　3. N 自由度系の基準振動

N 連成振子の運動方程式：

$$M\,\ddot{x_n} = -K(x_n - x_{n-1}) - K(x_n - x_{n+1}) \qquad (n = 1, 2, \cdots, N) \tag{1}$$

ただし，x_0, x_{N+1} は補助変数であり，次の条件を満たすものとする:

$$x_0 = 0, \qquad x_{N+1} = 0 \tag{2}$$

※) 条件 (2) は境界条件の一例．この場合，固定端条件に対応．

※) 運動方程式の並進対称性: 境界条件 (2) を無視して，振子が無限個つながっているものとすると，解 $x_n(t) = f_n(t)$ が得られれば，任意の整数 m について $x_n(t) = f_{n+m}(t)$ も解になる．

N 連成振子の基準振動：

$$\begin{aligned} x_n(t) &= A_{[m]}\sqrt{\frac{2}{N+1}}\sin\left[n\left(\frac{\pi}{(N+1)}m\right)\right]\cos(\omega_{[m]}t + \alpha_{[m]}) \\ &= \sqrt{\frac{2}{N+1}}\sin\left[n\left(\frac{\pi}{(N+1)}m\right)\right]\left\{a_{[m]}\cos(\omega_{[m]}t) + b_{[m]}\sin(\omega_{[m]}t)\right\} \end{aligned}$$

ただし，

$$\omega_{[m]}^2 = \frac{4K}{M}\sin^2\left(\frac{\pi}{2(N+1)}m\right) \quad (m = 1, 2, \cdots, N)$$

解法：

$$x_n(t) = [a\cos(n\theta) + b\sin(n\theta)]\,\cos(\omega t + \alpha)$$

とおく (この解の形は並進対称性を満たしていることに注意)：

$$\omega^2 = \frac{K}{M}2(1-\cos\theta) = \frac{4K}{M}\sin^2(\theta/2)$$

$$a = 0, \qquad [a\cos(N+1)\theta + b\sin(N+1)\theta] = 0$$

$$\therefore \quad a = 0, \qquad \theta = \frac{\pi}{(N+1)}m \quad (m = 1, 2, \cdots, N)$$

※) $m = 1, 2, \cdots, N$ 以外の θ の解は，$m = 1, 2, \cdots, N$ のいずれかの解に帰着する．

一般解：

$$x_n(t) = \sum_{m=1}^{N} A_{[m]}\sqrt{\frac{2}{N+1}}\sin\left[n\left(\frac{\pi}{(N+1)}m\right)\right]\cos(\omega_{[m]}t + \alpha_{[m]})$$

基準振動解の構造：

$$\begin{aligned} \boldsymbol{x}(t) &= \sum_{m=1}^{N} \boldsymbol{e}^{[m]} \, Q_{[m]}(t) \\ &= \sum_{m=1}^{N} \boldsymbol{e}^{[m]} \, A_{[m]} \, \cos(\omega_{[m]} t + \alpha_{[m]}) \end{aligned}$$

ただし，

$$\boldsymbol{e}^{[m]} = (e_1^{[m]}, e_2^{[m]}, \cdots, e_N^{[m]})^t, \qquad e_n^{[m]} = \sqrt{\frac{2}{N+1}} \sin\left[n \left(\frac{\pi}{(N+1)} m \right) \right]$$

$$\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} = \delta_{mm'}, \qquad \sum_{m=1}^{N} \boldsymbol{e}^{[m]} (\boldsymbol{e}^{[m]})^t = \mathbb{I}_{N \times N}.$$

初期条件：

$$\begin{aligned} \boldsymbol{x}(0) &= \sum_{m=1}^{N} \boldsymbol{e}^{[m]} \, a_{[m]} & \therefore \quad a_{[m]} &= \boldsymbol{e}^{[m]} \cdot \boldsymbol{x}(0) \\ \dot{\boldsymbol{x}}(0) &= \sum_{m=1}^{N} \boldsymbol{e}^{[m]} \, b_{[m]} \omega_{[m]} & \therefore \quad b_{[m]} &= \boldsymbol{e}^{[m]} \cdot \dot{\boldsymbol{x}}(0) \, \frac{1}{\omega_{[m]}} \end{aligned}$$

N=3 の場合：

$$\omega_{[m]}^2 = \frac{4K}{M}\sin^2\left(\frac{\pi}{8}m\right) \quad (m = 1, 2, 3)$$

$$e_n^{[m]} = \frac{1}{\sqrt{2}}\sin\left[n\left(\frac{\pi}{4}m\right)\right] \quad (n = 1, 2, 3)$$

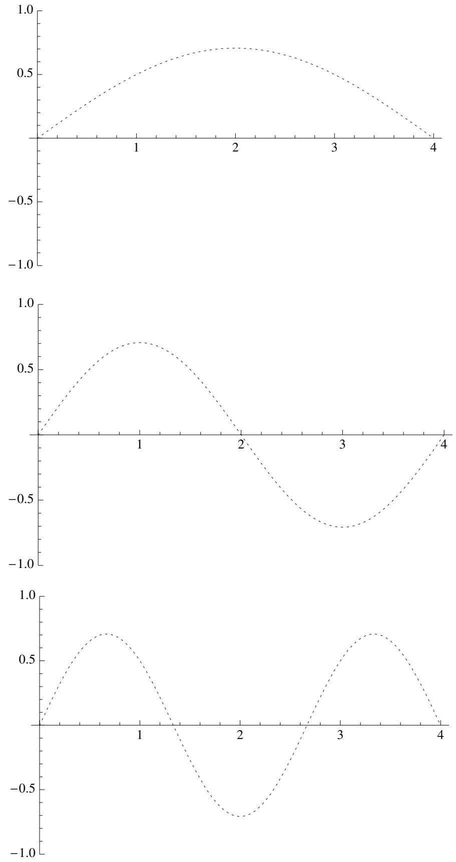

図 1: N=3

N=5 の場合：

$$\omega_{[m]}^2 = \frac{4K}{M}\sin^2\left(\frac{\pi}{12}m\right) \quad (m = 1, 2, 3, 4, 5)$$

$$e_n^{[m]} = \frac{1}{\sqrt{3}}\sin\left[n\left(\frac{\pi}{6}m\right)\right] \quad (n = 1, 2, 3, 4, 5)$$

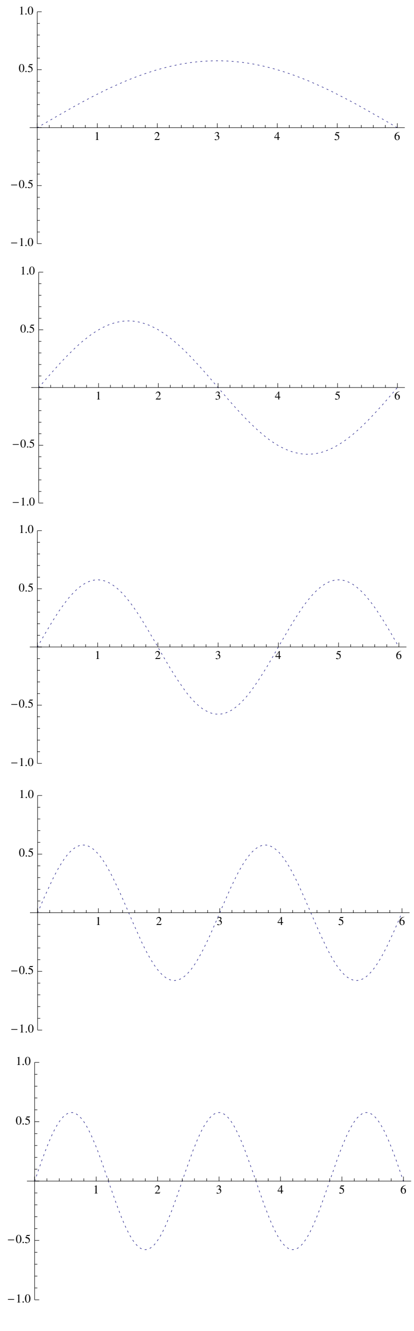

図 2: N=5

(計算メモ)

$$
\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} e_n^{[m]} e_n^{[m']} \\
&= \sum_{n=1}^{N} \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)} m\right)\right] \times \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)} m'\right)\right] \\
&= \sum_{n=1}^{N} \frac{2}{N+1} \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m}}{2i}\right) \times \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m'} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m'}}{2i}\right).
\end{aligned}
$$

ここで

$$
f(k) = \sum_{n=1}^{N} \mathrm{e}^{in\frac{\pi}{(N+1)}k}
$$

とおくと,

$$
\begin{aligned}
f(k) &= \sum_{n=1}^{N} \left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)^n \\
&= \mathrm{e}^{i\frac{\pi}{(N+1)}k} \frac{1 - \left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)^N}{1 - \left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)} \\
&= \frac{\mathrm{e}^{i\frac{\pi}{(N+1)}k} - (-1)^k}{1 - \left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)} = \begin{cases} N & k = 0 \\ -1 & k \neq 0,\ \text{偶数} \\ -f(-k) & k \neq 0,\ \text{奇数} \end{cases}
\end{aligned}
$$

を得る．これより

$$
\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} \frac{2}{N+1} \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m}}{2i}\right) \times \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m'} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m'}}{2i}\right) \\
&= -\frac{1}{2(N+1)} \left[f(m+m') + f(-m-m') - f(m-m') - f(-m+m')\right] \\
&= \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}
\end{aligned}
$$

同様にして

$$
\begin{aligned}
\sum_{m=1}^{N} \boldsymbol{e}^{[m]} \left(\boldsymbol{e}^{[m]}\right)^t &= \sum_{m=1}^{N} e_n^{[m]} e_{n'}^{[m]} \\
&= \sum_{m=1}^{N} \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)} m\right)\right] \times \sqrt{\frac{2}{N+1}} \sin\left[n'\left(\frac{\pi}{(N+1)} m\right)\right] \\
&= \begin{cases} 1 & n = n' \\ 0 & n \neq n' \end{cases}
\end{aligned}
$$