例題3+ (N 連成振り子の基準振動解の直交性，完全性)

例題３のＮ連成振り子の微小振動の基準振動解は次で与えられる：

1. 固定端（図１左）の場合

$$\boldsymbol{e}^{[m]} = (e_1^{[m]}, e_2^{[m]}, \cdots, e_N^{[m]})^t, \qquad e_n^{[m]} = \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)}m\right)\right]$$

2. 自由端（図１右）の場合

$$\boldsymbol{e}^{[m]} = (e_1^{[m]}, e_2^{[m]}, \cdots, e_N^{[m]})^t,$$
$$e_n^{[0]} = \sqrt{\frac{1}{N}}, \qquad e_n^{[m]} = \sqrt{\frac{2}{N}} \cos\left[(n-1/2)\left(\frac{\pi}{N}m\right)\right] \quad (m = 1, \cdots, N-1)$$

これらの基準振動解の直交性，完全性を示せ。

(解答例)

1. 固定端（図１左）の場合：

$$\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} e_n^{[m]} e_n^{[m']} \\
&= \sum_{n=1}^{N} \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)}m\right)\right] \times \sqrt{\frac{2}{N+1}} \sin\left[n\left(\frac{\pi}{(N+1)}m'\right)\right] \\
&= \sum_{n=1}^{N} \frac{2}{N+1} \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m}}{2i}\right) \times \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m'} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m'}}{2i}\right).
\end{aligned}$$

ここで

$$f(k) = \sum_{n=1}^{N} \mathrm{e}^{in\frac{\pi}{(N+1)}k}$$

とおくと，

$$\begin{aligned}
f(k) &= \sum_{n=1}^{N} \left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)^n \\
&= \mathrm{e}^{i\frac{\pi}{(N+1)}k} \frac{1-\left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)^N}{1-\left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)} \\
&= \frac{\mathrm{e}^{i\frac{\pi}{(N+1)}k} - (-1)^k}{1-\left(\mathrm{e}^{i\frac{\pi}{(N+1)}k}\right)} = \begin{cases} N & k = 0 \\ -1 & k \neq 0,\ \text{偶数} \\ -f(-k) & k \neq 0,\ \text{奇数} \end{cases}
\end{aligned}$$

を得る．これより

$$
\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} \frac{2}{N+1} \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m}}{2i} \right) \times \left(\frac{\mathrm{e}^{in\frac{\pi}{(N+1)}m'} - \mathrm{e}^{-in\frac{\pi}{(N+1)}m'}}{2i} \right) \\
&= -\frac{1}{2(N+1)} \left[f(m+m') + f(-m-m') - f(m-m') - f(-m+m') \right] \\
&= \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}
\end{aligned}
$$

同様にして

$$
\begin{aligned}
\sum_{m=1}^{N} \boldsymbol{e}^{[m]} \left(\boldsymbol{e}^{[m]} \right)^t &= \sum_{m=1}^{N} e_n^{[m]} e_{n'}^{[m]} \\
&= \sum_{m=1}^{N} \sqrt{\frac{2}{N+1}} \sin \left[n \left(\frac{\pi}{(N+1)} m \right) \right] \times \sqrt{\frac{2}{N+1}} \sin \left[n' \left(\frac{\pi}{(N+1)} m \right) \right] \\
&= \begin{cases} 1 & n = n' \\ 0 & n \neq n' \end{cases}
\end{aligned}
$$

2. 自由端の場合：

直交性：

$m, m' = 1, \cdots, N-1$:

$$\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} e_n^{[m]} e_n^{[m']} \\
&= \sum_{n=1}^{N} \sqrt{\frac{2}{N}} \cos\left[(n-1/2)\left(\frac{\pi}{N}m\right)\right] \times \sqrt{\frac{2}{N}} \cos\left[(n-1/2)\left(\frac{\pi}{N}m'\right)\right] \\
&= \sum_{n=1}^{N} \frac{2}{N} \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m}}{2}\right) \times \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m'} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m'}}{2}\right).
\end{aligned}$$

ここで

$$g(k) = \sum_{n=1}^{N} \mathrm{e}^{i(n-1/2)\frac{\pi}{N}k}$$

とおくと,

$$\begin{aligned}
g(k) &= \mathrm{e}^{i\frac{\pi}{2N}k} \sum_{n=0}^{N-1} \left(\mathrm{e}^{i\frac{\pi}{N}k}\right)^n \\
&= \mathrm{e}^{i\frac{\pi}{2N}k} \frac{1-\left(\mathrm{e}^{i\frac{\pi}{N}k}\right)^N}{1-\left(\mathrm{e}^{i\frac{\pi}{N}k}\right)} \\
&= -\frac{1-(-1)^k}{\mathrm{e}^{i\frac{\pi}{2N}k} - \mathrm{e}^{-i\frac{\pi}{2N}k}} = \begin{cases} N & k=0 \\ 0 & k \neq 0,\ \text{偶数} \\ -g(-k) & k \neq 0,\ \text{奇数} \end{cases}
\end{aligned}$$

を得る．これより

$$\begin{aligned}
\boldsymbol{e}^{[m]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} \frac{2}{N} \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m}}{2}\right) \times \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m'} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m'}}{2}\right) \\
&= \frac{1}{2N}\left[g(m+m') + g(-m-m') + g(m-m') + g(-m+m')\right] \\
&= \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}
\end{aligned}$$

$m = 0, m' \neq 0$ (または $m \neq 0, m =' 0$):

$$\begin{aligned}
\boldsymbol{e}^{[0]} \cdot \boldsymbol{e}^{[m']} &= \sum_{n=1}^{N} e_n^{[0]} e_n^{[m']} \\
&= \sum_{n=1}^{N} \sqrt{\frac{1}{N}} \times \sqrt{\frac{2}{N}} \cos\left[(n-1/2)\left(\frac{\pi}{N}m'\right)\right] \\
&= \sum_{n=1}^{N} \frac{\sqrt{2}}{N} \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m'} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m'}}{2}\right) \\
&= \frac{1}{\sqrt{2}N} \left(\mathrm{e}^{i\frac{\pi}{2N}m'} \frac{1-(-1)^{m'}}{1-\mathrm{e}^{i\frac{\pi}{N}m'}} + \mathrm{e}^{-i\frac{\pi}{2N}m'} \frac{1-(-1)^{m'}}{1-\mathrm{e}^{-i\frac{\pi}{N}m'}}\right) = 0.
\end{aligned}$$

$m, m' = 0$: 自明

完全性：

$$\begin{aligned}
\sum_{m=0}^{N-1} \boldsymbol{e}^{[m]} \left(\boldsymbol{e}^{[m]}\right)^t &= \frac{1}{N} + \sum_{m=1}^{N-1} e_n^{[m]} e_{n'}^{[m]} \\
&= \frac{1}{N} + \sum_{m=1}^{N-1} \sqrt{\frac{2}{N}} \cos\left[(n-1/2)\left(\frac{\pi}{N}m\right)\right] \times \sqrt{\frac{2}{N}} \cos\left[(n'-1/2)\left(\frac{\pi}{N}m\right)\right] \\
&= \frac{1}{N} + \sum_{m=1}^{N-1} \frac{2}{N} \left(\frac{\mathrm{e}^{i(n-1/2)\frac{\pi}{N}m} + \mathrm{e}^{-i(n-1/2)\frac{\pi}{N}m}}{2}\right) \times \left(\frac{\mathrm{e}^{i(n'-1/2)\frac{\pi}{N}m} + \mathrm{e}^{-i(n'-1/2)\frac{\pi}{N}m}}{2}\right) \\
&= \frac{1}{N} + \frac{1}{2N}\left[\bar{g}(n+n'-1) + \bar{g}(-n-n'+1) + \bar{g}(n-n') + \bar{g}(-n+n')\right] \\
&= \begin{cases} 1 & n = n' \\ 0 & n \neq n' \end{cases}
\end{aligned}$$

ただし，ここで

$$\bar{g}(k) = \sum_{m=1}^{N-1} \mathrm{e}^{ik\frac{\pi}{N}m}$$

とおき，次の結果を用いた.

$$\begin{aligned}
\bar{g}(k) &= \sum_{m=1}^{N-1} \left(\mathrm{e}^{ik\frac{\pi}{N}}\right)^m \\
&= -1 + \frac{1-\left(\mathrm{e}^{ik\frac{\pi}{N}}\right)^N}{1-\left(\mathrm{e}^{ik\frac{\pi}{N}}\right)} = \begin{cases} N-1 & k=0 \\ -1 & k \neq 0,\ \text{偶数} \\ \frac{1+\left(\mathrm{e}^{i\frac{\pi}{N}k}\right)}{1-\left(\mathrm{e}^{i\frac{\pi}{N}k}\right)} = -\bar{g}(-k) & k \neq 0,\ \text{奇数} \end{cases}
\end{aligned}$$