

Chapter 3

Instanton Solutions in Non-Abelian Gauge Theory

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3.1 Conventions

Gauge fields:

$$A_\mu = gA_\mu^a T^a \quad (3.1)$$

T^a = anti-hermitian generator

$$[T^a, T^b] = f^{abc} T^c$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = gF_{\mu\nu}^a T^a \quad (3.2)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \quad (3.3)$$

Normalization: We normalize the generator as

$$\text{Tr} T^a T^b = -\frac{1}{2} \delta^{ab} \quad (3.4)$$

In the case of $G = SU(2)$, this corresponds precisely to the choice

$$T^a = t^a \equiv \frac{\tau^a}{2i} \quad (3.5)$$

Covariant derivative:

$$D_\mu \equiv \partial_\mu + A_\mu \quad (3.6)$$

$$F_{\mu\nu} = [D_\mu, D_\nu] \quad (3.7)$$

Inner product notation: Sometimes we use the following inner product notation

$$(T^a, T^b) \equiv \delta^{ab} \quad (= -2\text{Tr} T^a T^b) \quad (3.8)$$

Euclidean action:

$$\begin{aligned} S_E &= \frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a = -\frac{1}{2g^2} \int d^4x \text{Tr}(F_{\mu\nu} F_{\mu\nu}) \\ &= \frac{1}{4g^2} \int d^4x (F_{\mu\nu}, F_{\mu\nu}) \geq 0 \end{aligned} \quad (3.9)$$

Indices and ϵ -tensor: To conform to some of the important literatures, we use the convention

$$\begin{aligned}\mu, \nu &= 0, 1, 2, 3 \\ \epsilon_{0123} &= 1\end{aligned}\tag{3.10}$$

Remark: If one uses the convention $\mu = 1 \sim 4$ and $\epsilon_{1234} = 1$, self-dual and anti-self-dual solutions are switched.

3.2 Decomposition $SO(4) = SU(2) \times SU(2)$ and Quaternions

In the following, the instanton for $G = SU(2)$ will play a fundamental role. This solution intertwines the gauge group and the spacetime symmetry group $SO(4)$, which can be decomposed as $SU(2) \times SU(2)$. This decomposition is intimately related to the **quaternion**, which will play a basic role in the ADHM construction.

3.2.1 Decomposition of $SO(4)$

□ $SO(4)$ and its generators:

$SO(4)$ rotation is expressed as

$$\begin{aligned} x'_\mu &= \Lambda_{\mu\nu} x_\nu \\ \Lambda^T \Lambda &= 1 \end{aligned}$$

Writing $\Lambda = \exp(\xi)$ and considering the infinitesimal transformation, we easily find that ξ is real 4×4 antisymmetric matrix. The standard basis for such antisymmetric matrices can be taken as $L_{\mu\nu}$ defined by (choosing a convenient overall sign)

$$(L_{\mu\nu})_{\rho\sigma} \equiv -(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \quad (3.11)$$

In other words, $L_{\mu\nu}$ has **-1 at the position (μ, ν) and 1 at (ν, μ)** and 0 for all the other elements. The non-vanishing commutator for these generators is of the form

$$[L_{\mu\nu}, L_{\nu\rho}] = L_{\rho\mu} \quad \text{no sum over } \nu \quad (3.12)$$

(For instance, $[L_{23}, L_{31}] = L_{12}$.) ξ can then be decomposed as (watch for the order of indices)

$$\xi = -\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu} \quad (3.13)$$

$L_{\mu\nu}$ is a generator of rotation in the μ - ν plane. For example, Rotation in the 1-2 plane is

$$\begin{aligned} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \theta L_{12}x \end{aligned} \quad (3.14)$$

□ **$SU(2) \times SU(2)$ decomposition:**

If we define I_i and K_i ($i = 1, 2, 3$) as

$$\begin{aligned} I_i &\equiv \frac{1}{2}\epsilon_{ijk}L_{jk} \\ K_i &\equiv L_{0i} \end{aligned}$$

they satisfy the commutation relations

$$\begin{aligned} [I_i, I_j] &= \epsilon_{ijk}I_k \\ [I_i, K_j] &= \epsilon_{ijk}K_k \\ [K_i, K_j] &= \epsilon_{ijk}I_k \end{aligned}$$

Now define the following combinations

$$J_i^\pm \equiv \frac{1}{2}(I_i \pm K_i) \quad (3.15)$$

Then, we find that they generate separately the **algebra** of $SU(2)$:

$$\begin{aligned} [J_i^\pm, J_j^\pm] &= \epsilon_{ijk}J_k^\pm \\ [J_i^\pm, J_j^\mp] &= 0 \end{aligned}$$

To see that they generate really the **group** $SU(2) \times SU(2)$, compute the exponent $-\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu}$:

$$\begin{aligned} -\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu} &= -\xi_{0i}K_i - \frac{1}{2}\xi_{ij}\epsilon_{ijk}I_k \\ &= \left(-\frac{1}{2}\xi_{ij}\epsilon_{ijk} - \xi_{0k}\right)J_k^+ + \left(-\frac{1}{2}\xi_{ij}\epsilon_{ijk} + \xi_{0k}\right)J_k^- \\ &\equiv \theta_k^+ J_k^+ + \theta_k^- J_k^- \end{aligned}$$

Since θ_k^\pm are **real and independent**, the decomposition is indeed $SU(2) \times SU(2)$. In this regard, recall that for the Lorentz group, they are complex conjugate of each other.

□ **Intertwiner:**

We will need a more explicit relation between $SO(4)$ and $SU(2) \times SU(2)$.

Let M_{AB} be an $SU(2)$ transformation matrix and u_B be the fundamental spinor representation:

$$u'_A = M_{AB}u_B, \quad M^\dagger M = 1, \quad A, B = 1, 2 \quad (3.16)$$

We will use dotted indices such as \dot{A}, \dot{B} for the second $SU(2)$.

The above decomposition means that there must exist a 4×4 **intertwining matrix** $T_{A\dot{B},\mu}$ such that

$$\begin{aligned} T J_i^+ T^{-1} &\equiv \mathcal{J}_i^+ = \frac{\Sigma_i^+}{2i} \otimes \mathbf{1} \\ T J_i^- T^{-1} &\equiv \mathcal{J}_i^- = \mathbf{1} \otimes \frac{\Sigma_i^-}{2i} \end{aligned}$$

or more explicitly

$$\begin{aligned} T_{A\dot{B},\mu}(J_i^+)_{\mu\nu} &= \left(\frac{\Sigma_i^+}{2i} \otimes \mathbf{1}\right)_{A\dot{B},C\dot{D}} T_{C\dot{D},\nu} \\ T_{A\dot{B},\mu}(J_i^-)_{\mu\nu} &= \left(\mathbf{1} \otimes \frac{\Sigma_i^-}{2i}\right)_{A\dot{B},C\dot{D}} T_{C\dot{D},\nu} \end{aligned}$$

where Σ_i^\pm are 2×2 $SU(2)$ generators. A solution to these set of equations is, regarding $T_{A\dot{B},\mu}$ as four 2×2 matrices,

$$T_0 = 1, \quad T_i = -i\tau_i \quad (3.17)$$

$$\Sigma_i^+ = \tau_i, \quad \Sigma_i^- = -\tau_i^T \quad (3.18)$$

where τ_i are the Pauli matrices. Indeed Σ_i^\pm satisfy the same algebra. Hereafter, we shall write

$$\sigma_\mu \equiv T_\mu = (1, -i\tau_i) \quad (3.19)$$

In this way, we obtain the following explicit decomposition formula for general $SO(4)$ transformation:

$$x' = \Lambda x = e^{\theta_k^+ J_k^+} e^{\theta_k^- J_k^-} x \quad (3.20)$$

$$\begin{aligned} Tx' &= \sigma_\mu \Lambda_{\mu\nu} x_\nu = \left[T e^{\theta_k^+ J_k^+} T^{-1} \right] \left[T e^{\theta_k^- J_k^-} T^{-1} \right] Tx \\ &= \left[e^{\theta_k^+ \Sigma^+ / 2i} \otimes e^{\theta_k^- \Sigma^- / 2i} \right] Tx \\ &= e^{\theta_k^+ \Sigma^+ / 2i} \sigma_\nu e^{\theta_k^- (\Sigma^-)^T / 2i} x_\nu \end{aligned} \quad (3.21)$$

Removing x_ν , this can be written as

$$\sigma_\mu \Lambda_{\mu\nu} = M_+ \sigma_\nu M_-^T \quad (3.22)$$

$$M_\pm = SU(2)_\pm \quad (3.23)$$

Therefore, σ_μ transforms under bifundamental $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$.

3.2.2 Quaternions

σ_μ defined above is deeply related to the **quaternions**, which forms an algebra (actually a field) denoted by **H**.

Quaternion $q \in \mathbf{H}$ is defined by

$$q = \sum_{\mu=0}^3 q_{\mu} e_{\mu} = q_0 e_0 + \sum_{i=1}^3 q_i e_i \quad (3.24)$$

where q_{μ} are **real** numbers and e_{μ} 's satisfy the following closed algebra:

$$e_0 e_0 = e_0, \quad e_i e_i = -e_0, \quad (3.25)$$

$$e_0 e_i = e_i e_0 = e_i, \quad e_i e_j = \epsilon_{ijk} e_k \quad (i \neq j) \quad (3.26)$$

Since $e_i e_j = -e_j e_i$, quaternion algebra is in general **non-commutative**. This multiplication rule can be summarized as (μ, ν on the RHS are not summed)

$$e_{\mu} e_{\nu} = -(-1)^{\delta_{\mu 0}} \delta_{\mu\nu} e_0 + \delta_{\mu 0} (1 - \delta_{\nu 0}) e_{\nu} + \delta_{\nu 0} (1 - \delta_{\mu 0}) e_{\mu} + \sum_{\rho} \epsilon_{0\mu\nu\rho} e_{\rho} \quad (3.27)$$

When one sets $e_2 = e_3 = 0$, then it becomes a complex number, with e_1 being the imaginary unit i . Hereafter summation over the repeated indices will be assumed, unless otherwise stated.

It is clear that \mathbf{H} is closed under multiplication.

Remark: One can easily check that if we ignore the fact that σ_{μ} has the index structure $(\sigma_{\mu})_{A\dot{B}}$, namely that row and column indices are acted on by different $SU(2)$ groups, σ_{μ} satisfies exactly the same algebra as quaternions defined above.

□ **Conjugation, (anti-)self-duality and norm:**

Quaternionic conjugate of q will be denoted by q^{\dagger} and is defined by

$$q^{\dagger} \equiv q_{\mu} e_{\mu}^{\dagger} \quad (3.28)$$

$$e_0^{\dagger} = e_0, \quad e_i^{\dagger} = -e_i \quad (3.29)$$

Consider now the products $e_\mu e_\nu^\dagger$ and $e_\mu^\dagger e_\nu$. Due to the following group theoretical reasons, they have very simple interpretations:

If we go back to σ_μ interpretation of quaternions, $e_\mu e_\nu^\dagger$ is a quantity which transforms solely by $SU(2)_+$, like $M_+ e_\mu e_\nu^\dagger M_+^\dagger$. Thus, it is a singlet under $SU(2)_-$ and is therefore a mixture of $(0, 0)$ and $(1, 0)$:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0)$$

The former must be represented by $\delta_{\mu\nu}$ and the latter should be a **self-dual** antisymmetric tensor, which will be denoted by $2i\sigma_{\mu\nu}$.

Similarly, $e_\mu^\dagger e_\nu$ transforms under $SU(2)_-$ like $M_-^* e_\mu^\dagger e_\nu M_-^T$ and is a mixture of $(0, 0)$ and $(0, 1)$. The latter is the **anti-self-dual** part of $SO(4)$ tensor.

In fact, one easily verifies the following relations:

$$e_\mu e_\nu^\dagger = \delta_{\mu\nu} + 2i\sigma_{\mu\nu} \quad (3.30)$$

$$e_\mu^\dagger e_\nu = \delta_{\mu\nu} + 2i\bar{\sigma}_{\mu\nu} \quad (3.31)$$

$$\sigma_{\mu\nu} = \frac{1}{4i}(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) \quad (3.32)$$

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4i}(e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu) \quad (3.33)$$

where $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are antisymmetric, hermitian and satisfy the

following **duality properties**:

$$*\sigma_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = \sigma_{\mu\nu} \quad \text{self-dual(SD)}$$

$$*\bar{\sigma}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = -\bar{\sigma}_{\mu\nu} \quad \text{anti-self-dual(ASD)}$$

where $\epsilon_{0123} \equiv 1$

(For example, $*(e_0e_1^\dagger) = e_2e_3^\dagger = -e_1 = e_0e_1^\dagger$ satisfying self-duality.)

't Hooft tensor: $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are traceless as 2×2 matrices. Thus, they can be expanded in terms of e_a , with $a = 1, 2, 3$. Explicitly,

$$\sigma_{\mu\nu} \equiv \frac{1}{2}\eta_{\mu\nu}^a (ie_a) = \eta_{\mu\nu}^a \frac{\tau_a}{2} \quad (3.34)$$

$$\eta_{\mu\nu}^a = \delta_{\mu 0}\delta_{\nu a} - \delta_{\nu 0}\delta_{\mu a} + \epsilon_{0a\mu\nu} \quad (3.35)$$

$$\bar{\sigma}_{\mu\nu} \equiv \frac{1}{2}\bar{\eta}_{\mu\nu}^a (ie_a) = \bar{\eta}_{\mu\nu}^a \frac{\tau_a}{2} \quad (3.36)$$

$$\bar{\eta}_{\mu\nu}^a = -(\delta_{\mu 0}\delta_{\nu a} - \delta_{\nu 0}\delta_{\mu a}) + \epsilon_{0a\mu\nu} \quad (3.37)$$

$\eta_{\mu\nu}^a, \bar{\eta}_{\mu\nu}^a$ are often called **'t Hooft tensors**. In the construction of instanton solution, e_a will be regarded as the basis for the gauge group $SU(2)$. Thus, 't Hooft tensors intertwine the spacetime and internal groups.

Properties of the 'tHooft tensors:

$$\eta_{a\mu\nu} = \epsilon_{a\mu\nu}, \quad \text{if } \mu, \nu = 1, 2, 3$$

$$\eta_{a0\nu} = \delta_{a\nu}$$

$$\eta_{a\mu 0} = -\delta_{a\mu}$$

$$\eta_{a00} = 0$$

$$\bar{\eta}_{a\mu\nu} = (-1)^{\delta_{\mu 0} + \delta_{\nu 0}} \eta_{a\mu\nu}$$

$$\begin{aligned}
\eta_{a\mu\nu}\eta_{b\mu\nu} &= 4\delta_{ab} \\
\eta_{a\mu\nu}\eta_{a\mu\lambda} &= 3\delta_{\nu\lambda} \\
\eta_{a\mu\nu}\eta_{a\mu\nu} &= 12 \\
\eta_{a\mu\nu}\eta_{a\kappa\lambda} &= \delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa} + \epsilon_{\mu\nu\kappa\lambda} \\
\delta_{\kappa\lambda}\eta_{a\mu\nu} + \delta_{\kappa\nu}\eta_{a\lambda\mu} + \delta_{\kappa\mu}\eta_{a\nu\lambda} + \eta_{a\sigma\kappa}\epsilon_{\lambda\mu\nu\sigma} &= 0 \\
\eta_{a\mu\nu}\eta_{b\mu\lambda} &= \delta_{ab}\delta_{\nu\lambda} + \epsilon_{abc}\eta_{c\nu\lambda} \\
\epsilon_{abc}\eta_{b\mu\nu}\eta_{c\kappa\lambda} &= \delta_{\mu\kappa}\eta_{a\nu\lambda} - \delta_{\mu\lambda}\eta_{a\nu\kappa} - \delta_{\nu\kappa}\eta_{a\mu\lambda} + \delta_{\nu\lambda}\eta_{a\mu\kappa} \\
\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} &= 0 \\
\eta_{a\kappa\mu}\bar{\eta}_{b\kappa\lambda} &= \eta_{a\kappa\lambda}\bar{\eta}_{b\kappa\mu}
\end{aligned}$$

Norm: The norm-squared of q is defined as $q^\dagger q$:

$$|q|^2 = q^\dagger q = q_\mu q_\nu e_\mu^\dagger e_\nu = q_\mu q_\nu \delta_{\mu\nu} = \sum q_\mu q_\mu \geq 0 \quad (3.38)$$

where the equality holds if and only if $q = 0$. It is clear that $q^\dagger q = qq^\dagger$ holds. Furthermore, this shows that non-vanishing q always has the inverse q^{-1} (hence \mathbf{H} forms a field) given by

$$q^{-1} = \frac{q^\dagger}{|q|^2}, \quad qq^{-1} = q^{-1}q = 1 \quad (3.39)$$

□ **Another view of 't Hooft tensor:**

The 't Hooft tensor can be introduced from a slightly different point of view. The idea is to extend the action of $SO(4)$ to the internal gauge group part. One natural way is to **intertwine the gauge group $SU(2)_g$ with one of the $SU(2)$ factor of $SO(4)$.**

For instance, extend $SU(2)_+$ to $SU(2)_+ \oplus SU(2)_g$. The total generator for this sector becomes

$$\tilde{J}_i^+ = J_i^+ + t_i \quad (3.40)$$

where t_i is the generator of $SU(2)_g$ satisfying $[t_i, t_j] = \epsilon_{ijk}t_k$. $SU(2)_-$ sector is unchanged, i.e. $\tilde{J}_i^- = J_i^-$. Then, going backwards to $L_{\mu\nu}$, we easily find the following modified expressions denoted by $\tilde{L}_{\mu\nu}$:

$$\begin{aligned} \tilde{L}_{\mu\nu} &= L_{\mu\nu} + l_{\mu\nu} \\ \text{where } l_{ij} &= \epsilon_{ijk}t_k \\ l_{0i} &= t_i \end{aligned}$$

Since \tilde{J}_i^\pm satisfy exactly the same commutation relations as before, $\tilde{L}_{\mu\nu}$ (and hence $l_{\mu\nu}$ themselves) obey $SO(4)$ algebra. Now introduce $\eta_{a\mu\nu}$ by

$$l_{\mu\nu} = \eta_{a\mu\nu}t_a = \eta_{a\mu\nu}\frac{\tau_a}{2i} \quad (3.41)$$

Then, one can check that $\eta_{a\mu\nu}$ is exactly the 't Hooft tensor. This also means the identification $\sigma_{\mu\nu} = i l_{\mu\nu}$.

□ **Conversion between e_μ and e_μ^\dagger :**

The following relation is often useful:

$$\begin{aligned} \epsilon e_\mu \epsilon^T &= e_\mu^*, \quad \epsilon = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \Leftrightarrow \epsilon_{AA'}(e_\mu)_{A'B'}\epsilon_{B'B} &= -(e_\mu^\dagger)_{BA} \end{aligned}$$

3.3 (Anti-)Self-Dual Configurations as Classical Solutions

3.3.1 Some formulas

We define the dual field strength as

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (3.42)$$

Then we find

$$\mathbf{F}_{\mu\nu}^2 = \tilde{\mathbf{F}}_{\mu\nu}^2 \quad (3.43)$$

Proof:

$$\tilde{\mathbf{F}}_{\mu\nu}^2 = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} = \frac{1}{4}2(\delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho})F_{\alpha\beta}F_{\rho\sigma} = F_{\mu\nu}^2$$

Using this formula, we get (with $F = F_{\mu\nu}^a$ etc.)

$$\frac{1}{2}(F \pm \tilde{F})^2 = \frac{1}{2}(F^2 + \tilde{F}^2 \pm 2F\tilde{F}) = F^2 \pm F\tilde{F} \geq 0 \quad (3.44)$$

Since $F^2 \geq 0$, this implies

$$\mathbf{F}^2 \geq |\mathbf{F}\tilde{\mathbf{F}}| \quad (3.45)$$

where the equality holds when $F \pm \tilde{F} = 0$, *i.e.* for $\mathbf{F}_{\mu\nu}^a = \pm \tilde{\mathbf{F}}_{\mu\nu}^a$. These are called **self-dual (SD)** and **anti-self-dual (ASD)** configurations. Hereafter, we use (A)SD to denote both of these configurations.

3.3.2 Minimum action configurations

Integrate the relation (3.44) above over the space-time. We get

$$S_E = \frac{1}{4g^2} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a \geq |Q| \quad (3.46)$$

$$\text{where } Q \equiv \frac{1}{4g^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \quad (3.47)$$

and the minimum value of the action is attained for (A)SD configurations.

□ **Equation of motion and Bianchi identity:**

(A)SD configurations are necessarily **solutions** of the classical YM equation

$$[D_\mu, F_{\mu\nu}] = 0 \quad (3.48)$$

This is because the Bianchi identity

$$0 = [D_\mu, \tilde{F}_{\mu\nu}] = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} [D_\mu, [D_\alpha, D_\beta]] \quad (3.49)$$

is equivalent to the equation of motion for (A)SD configurations.

□ **Vanishing of the energy-momentum tensor:**

The energy-momentum tensor is given by (omitting the group theory superscript a)

$$T_{\mu\nu} = F_{\mu\lambda}F_{\lambda\nu} - \frac{1}{4}\delta_{\mu\nu}(F_{\alpha\beta}F_{\beta\alpha}) \quad (3.50)$$

Exercise: Prove that $T_{\mu\nu} = 0$ for (A)SD configurations.

Now we use the following identity:

$$\tilde{F}_{\mu\lambda}\tilde{F}_{\lambda\nu} = \frac{1}{2}\delta_{\mu\nu}F_{\alpha\beta}F_{\beta\alpha} - F_{\mu\alpha}F_{\alpha\nu} \quad (3.51)$$

This can be proved by direct calculation:

$$\begin{aligned} \tilde{F}_{\mu\lambda}\tilde{F}_{\lambda\nu} &= \frac{1}{4}\epsilon_{\mu\lambda\alpha_1\alpha_2}\epsilon_{\lambda\nu\beta_1\beta_2}F_{\alpha_1\alpha_2}F_{\beta_1\beta_2} \\ &= -\frac{1}{4}\begin{vmatrix} \delta_{\mu\nu} & \delta_{\mu\beta_1} & \delta_{\mu\beta_2} \\ \delta_{\alpha_1\nu} & \delta_{\alpha_1\beta_1} & \delta_{\alpha_1\beta_2} \\ \delta_{\alpha_2\nu} & \delta_{\alpha_2\beta_1} & \delta_{\alpha_2\beta_2} \end{vmatrix} F_{\alpha_1\alpha_2}F_{\beta_1\beta_2} \\ &= \dots \\ &= \frac{1}{2}\delta_{\mu\nu}F_{\alpha\beta}F_{\beta\alpha} - F_{\mu\alpha}F_{\alpha\nu} \end{aligned} \quad (3.52)$$

Therefore, we get

$$\frac{1}{4}\delta_{\mu\nu}(F_{\alpha\beta}F_{\beta\alpha}) = \frac{1}{2}(F_{\mu\alpha}F_{\alpha\nu} + \tilde{F}_{\mu\alpha}\tilde{F}_{\alpha\nu}) \quad (3.53)$$

Putting this into $T_{\mu\nu}$, we find

$$T_{\mu\nu} = \frac{1}{2}(F_{\mu\alpha}F_{\alpha\nu} - \tilde{F}_{\mu\alpha}\tilde{F}_{\alpha\nu}) \quad (3.54)$$

which obviously vanishes for (A)SD configurations.

3.4 Winding Number for Finite Action Configurations

3.4.1 Topological nature of the charge Q

It is easy to see that Q is topological in the sense that it is invariant under any continuous deformation of A_μ . In fact

$$\begin{aligned}\delta\text{Tr}F_{\mu\nu}\tilde{F}_{\mu\nu} &= 2\text{Tr}(\delta[D_\mu, D_\nu]\tilde{F}_{\mu\nu}) \\ &= 4\text{Tr}([\partial_\mu + A_\mu, \delta A_\nu]\tilde{F}_{\mu\nu}) \\ &= \partial_\mu\text{Tr}(4\delta A_\nu\tilde{F}_{\mu\nu}) + 4\text{Tr}(\delta A_\nu[D_\mu, \tilde{F}_{\mu\nu}])\end{aligned}\quad (3.55)$$

Due to the Bianchi identity, the second term vanishes and the result is a total derivative. Upon integration this vanishes if at least $F_{\mu\nu}$ tends to zero at infinity. Note that this is true for any configuration.

Remark: We have already discussed this in the lecture on anomaly. There we derived the formula

$$\delta\text{Tr}(F^{n+1}) = d((n+1)\text{Tr}(\delta AF^n)) \quad (3.56)$$

For $n = 1$, this is nothing but the above equation:

$$\begin{aligned}F^2 &= F \wedge F = \frac{1}{2}F_{\mu\nu}\tilde{F}_{\mu\nu}d^4x \\ \delta AF &= \delta A \wedge F = \frac{1}{2}A_\mu F_{\alpha\beta}dx^\mu dx^\alpha dx^\beta \\ \therefore d(\delta AF) &= \partial_\nu(\delta A_\mu\tilde{F}_{\nu\mu})d^4x\end{aligned}$$

In fact, we showed that $\text{Tr}F^2 = d\omega_3^0$, where

$$\omega_3^0 = \text{Tr}\left(AF - \frac{1}{3}A^3\right) = \text{Tr}\left(AdA + \frac{2}{3}A^3\right) = \text{Chern-Simons form}$$

More explicitly,

$$d\omega_3^0 = \partial_\mu K_\mu d^4x$$

$$K_\mu = \epsilon_{\mu\nu\alpha\beta} \text{Tr} \left(\frac{1}{2} A_\nu F_{\alpha\beta} - \frac{1}{3} A_\nu A_\alpha A_\beta \right)$$

3.4.2 Non-trivial gauge transformations and their winding number

For the **action to be finite**, $F_{\mu\nu}$ must fall off faster than $1/r^2$ as $r = \sqrt{x^2} \rightarrow \infty$. This means that **the gauge potential must fall off faster than $1/r$ up to a gauge transformation**, *i.e.*

$$A_\mu \sim g^{-1} \partial_\mu g + o\left(\frac{1}{r}\right)$$

□ **Case of $G = SU(2)$:**

The most general $SU(2)$ gauge transformation can be written as

$$g = a + ib_i \tau_i$$

where a, b_i ($i = 1, 2, 3$) are real numbers satisfying

$$a^2 + b_i b_i = 1 \quad \Leftrightarrow \quad g^\dagger g = 1$$

This shows that $SU(2)$ is topologically a 3-sphere S^3 .

Note that **g is nothing but a quaternion q with the unit norm $q^\dagger q = 1$** . This is the well-known equivalence

$$SU(2) \simeq Sp(1)$$

When a and b_i become functions of x_μ , $g(x)$ for large fixed r gives a mapping $S^3(\text{space time}) \rightarrow S^3(SU(2))$. The important fact is that such gauge transformations are classified by the

homotopy group, *i.e.* the additive group of equivalence class whose members are continuously deformable to each other.

The simplest non-trivial gauge transformation which is not homotopic to a constant can be represented by

$$g = \frac{1}{r}(x_0 + i\vec{x} \cdot \vec{\tau}) = \frac{1}{r}\sigma_\nu^\dagger x_\nu$$

It is clear that, for a fixed r , as one covers S^3 in space one covers $SU(2)$ group space exactly once.

Let us compute the **pure gauge potential corresponding to this g** :

$$\begin{aligned} g^{-1} &= \frac{1}{r}\sigma_\nu x_\nu \\ \partial_\mu g &= \frac{1}{r}\sigma_\mu^\dagger - \frac{x_\mu}{r^3}\sigma_\nu^\dagger x_\nu \\ A_\mu &= g^{-1}\partial_\mu g = \frac{1}{r}\sigma_\lambda x_\lambda \left(\frac{1}{r}\sigma_\mu^\dagger - \frac{x_\mu}{r^3}\sigma_\nu^\dagger x_\nu \right) = \dots \\ &= \frac{-2i\sigma_{\mu\lambda} x_\lambda}{r^2} = \frac{2l_{\mu\lambda} x_\lambda}{r^2} \end{aligned}$$

where we used $\sigma_\mu \sigma_\nu^\dagger = \delta_{\mu\nu} + 2i\sigma_{\mu\nu}$.

Since, as we have seen, the topological charge is a homotopy invariant, it must characterize the **homotopy class** of the gauge transformation g . Since except at the origin F vanishes for a pure gauge potential, we have

$$\begin{aligned} \int \text{Tr} F^2 &= \int d^4x \partial_\mu K_\mu^g = \int_{S^3} d^3x \frac{x_\mu}{r} K_\mu^g \\ K_\mu^g &= -\frac{1}{3}\epsilon_{\mu\nu\alpha\beta} \text{Tr}(A_\nu A_\alpha A_\beta) \end{aligned}$$

(AF part of K_μ is zero for pure gauge configuration.)

To evaluate this, note that

- (i) K_μ^g is a vector and hence $x_\mu K_\mu^g$ is rotationally invariant and
- (ii) we may set $r = 1$ since we are computing a homotopy invariant.

Thus, all we have to do is to compute the value of the integrand **at one point on a unit S^3 and multiply by the volume of S^3** , which is $2\pi^2$. Take the point to be $x_0 = 1, x_i = 0$. Then,

$$\begin{aligned} A_i &= -2i\sigma_{i0} = i\tau_i \\ \frac{x_\mu}{r} K_\mu^g &= K_0^g = \frac{i}{3} \epsilon_{ijk} \text{Tr}(\tau_i \tau_j \tau_k) = -\frac{2}{3} \epsilon_{ijk} \epsilon_{ijk} = -4 \end{aligned}$$

Thus, if we define the **Pontryagin index (or winding number) k** by

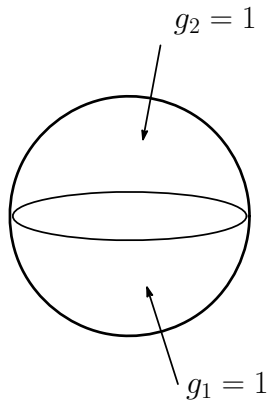
$$k \equiv \frac{1}{32\pi^2} \int d^4x F_\mu^a \tilde{F}_{\mu\nu}^a = -\frac{1}{8\pi^2} \int \text{Tr} F^2$$

we get

$$k = -\frac{1}{8\pi^2} (-4) 2\pi^2 = 1$$

for the above homotopy class.

Additivity of the winding number: For more general gauge transformation, the following observation suffices: Let the winding number of $g_i, i = 1, 2$ be k_i and consider the product $g = g_1 g_2$.



Since the winding number is unchanged by continuous deformation, we may deform $g_1(g_2)$ such that $g_1 = 1(g_2 = 1)$ on the lower (upper) hemisphere of S^3 . In this case the winding number k_1 for g_1 is obtained by integration over the upper hemisphere only and so on. It is then clear that the winding number of g_1g_2 is $k_1 + k_2$.

3.5 One Instanton Solution for $SU(2)$

With these preparations, we now describe how to obtain the simplest (anti-)instanton solution with $k = \pm 1$.

Since the (A)SD equations are still rather difficult to solve in complete generality, one would like to make an **ansatz** to find solutions.

The most natural strategy is to first look for a **self-dual solution with $SO(4)$ symmetry**. An obvious ansatz (adopted by BPST) to try is

$$A_\mu = g A_\mu^a t_a = \eta_{\mu\nu}^a t_a x_\nu f(x^2) = l_{\mu\nu} x_\nu f(x^2)$$

which satisfies the gauge condition $x_\mu A_\mu = 0$. Using the $SO(4)$ commutation relations, we can easily compute $F_{\mu\nu}$ to be

$$F_{\mu\nu} = \underbrace{l_{\mu\nu}}_{SD} (x^2 f^2 - 2f) + \underbrace{(x_\mu l_{\nu\lambda} x_\lambda - (\mu \leftrightarrow \nu))}_{ASD} (2f' + f^2)$$

where f' means derivative with respect to x^2 . For this to be self-dual, we must set $2f' + f^2 = 0$. The general solution of this equation is

$$f(x^2) = \frac{2}{x^2 + \rho^2} \quad (3.57)$$

with ρ a constant. Thus we get **a regular self-dual solution**

$$A_\mu = \frac{2l_{\mu\nu}x_\nu}{x^2 + \rho^2}, \quad A_\mu^a = \frac{2}{g} \frac{\eta_{\mu\nu}^a x_\nu}{x^2 + \rho^2} \quad (3.58)$$

$$F_{\mu\nu} = -\frac{4l_{\mu\nu}\rho^2}{(x^2 + \rho^2)^2}, \quad F_{\mu\nu}^a = -\frac{4}{g} \frac{\eta_{\mu\nu}^a \rho^2}{(x^2 + \rho^2)^2} \quad (3.59)$$

- ρ can be interpreted as the **size** of the instanton.
- From translation invariance, we may replace x_μ by $x_\mu - a_\mu$ with \mathbf{a}_μ describing the **position** of the instanton.
- Thus, this solution has **5 gauge-invariant free parameters**, called the **moduli** of an instanton solution.
- With $\bar{\eta}_{\mu\nu}^a$ replacing $\eta_{\mu\nu}^a$, one gets the anti-instanton solution.

Note that as $r \rightarrow \infty$, A_μ precisely reduces to the pure gauge $g^{-1}\partial_\mu g$ carrying winding number 1, with g discussed previously.

3.6 A Class of Multi-Instanton Solutions

3.6.1 Extended Ansatz

A more general ansatz which yields a class of multi-instanton solutions is of the form¹

$$A_\mu = l_{\mu\alpha}\partial_\alpha f(\mathbf{x}) \quad (3.60)$$

which satisfies the gauge condition $\partial_\mu A_\mu = 0$. One can easily compute $F_{\mu\nu}$ to be

$$F_{\mu\nu} = l_{\mu\nu}(\partial f)^2 - l_{\mu\rho}S_{\rho\nu} + l_{\nu\rho}S_{\rho\mu}$$

where $S_{\mu\nu} = \partial_\nu\partial_\nu f + \partial_\mu f\partial_\nu f = \text{symmetric}$

Now decompose $V_{\alpha\beta}$ into the traceless part and the trace part:

$$S_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}\delta_{\mu\nu}S$$

$$S = S_{\mu\mu}, \quad T_{\mu\mu} = 0$$

Then, we get

$$F_{\mu\nu} = \frac{1}{2}l_{\mu\nu}((\partial f)^2 - \partial^2 f) + A_{\mu\nu}$$

$$A_{\mu\nu} = -l_{\mu\rho}T_{\rho\nu} + l_{\nu\rho}T_{\rho\mu}$$

The first term is clearly self-dual. Although it is not at all obvious, $A_{\mu\nu}$ part is actually anti-self-dual. This can be checked by studying \tilde{A}_{i0} and \tilde{A}_{ij} separately. The reason for it is

¹F. Wilczek, in "Quark confinement and Field Theory", ed. D. Stump and D. Wein-garten (New York, 1977); E. Corrigan and D.B. Fairlie, PLB67 (77) 69.

roughly as follows: In terms of $SU(2) \times SU(2)$ representations, $l_{\mu\nu} \in (1, 0)$ and $T_{\mu\nu}$ (traceless, symmetric) $\in (1, 1)$. Thus $(1, 0) \times (1, 1) = (0, 1) \oplus (2, 1)$. The above combination picks up the anti-self-dual part $(0, 1)$.

3.6.2 Self-dual solution

Self-dual solution is obtained if we set $A_{\mu\nu} = 0$, *i.e.* $T_{\mu\nu} = 0$:

$$S_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}S = \partial_\mu\partial_\nu f + \partial_\mu f\partial_\nu f - \frac{1}{4}\delta_{\mu\nu}(\partial^2 f + (\partial f)^2) = 0$$

It is convenient to set $\mathbf{f} = -\ln \varphi$. Then, the equation above can be rewritten as

$$\partial_\mu \left(\frac{\partial_\nu \varphi}{\varphi^2} \right) = \frac{1}{4}\delta_{\mu\nu}\partial_\rho \left(\frac{\partial_\rho \varphi}{\varphi^2} \right)$$

This means that $\partial_\nu \varphi / \varphi^2$ can only be a linear function of x_ν of the form $cx_\nu + d_\nu$. Thus, we have

$$\frac{\partial_\nu \varphi}{\varphi^2} = \partial_\nu(-\varphi^{-1}) = cx_\nu + d_\nu$$

For $c \neq 0$, the solution is of the form

$$\varphi = -\frac{1}{\frac{1}{2}c(x-a)^2 + b}$$

This gives a **finite action only if the sign of c and b are the same, so that φ never blows up**. In such a case, it coincides with the BPST solution, which is not new. For $c = 0$, $F_{\mu\nu}$ becomes singular.

3.6.3 Anti-self-dual solution

Another possibility is to set the self-dual part to zero. This will turn out to give **more interesting solutions**. The equation

is $(\partial f)^2 = \partial^2 f$ and just as before, set $f = -\ln \varphi$. Then this simplifies to

$$\frac{\partial^2 \varphi}{\varphi} = 0$$

The most general solution with positive definite sign with isolated singularities is

$$\varphi = \sum_{i=1}^N \frac{\rho_i^2}{(x - a_i)^2} + c^2$$

- Due to the division by φ , the δ -function is annihilated and this is a legitimate solution of the above equation.
- Since the equation above is defined **only up to an overall constant**, there are actually only **two types of solutions**: **$c = 1$** (first considered by 't Hooft) and **$c = 0$** (introduced by Jackiw, Nohl and Rebbi).

As we shall see, **they represent multi-anti-instanton solutions with winding number $-N$ and $-(N - 1)$** respectively.

3.6.4 Regular Solution by Gauge Transformations

The ASD solution above is singular at N points $x = a_i$. Actually, these singularities are **gauge artifacts**.

To show this, we must be rather careful and **define the gauge field B_μ which is equal to A_μ except at the singular points.** Explicitly,

$$B_\mu = -\frac{2i\sigma_{\mu\nu}}{\varphi} \sum_{i=1}^N \frac{\rho_i^2 (x - a_i)_\nu}{(x - a_i)^4}, \quad x \neq a_i$$

(This form is exactly what we get if we formally compute A_μ .)

$c = 0$ case: Consider first the $c = 0$ case. Then, it has the following asymptotic behavior as $x \rightarrow \infty$:

$$\begin{aligned} \varphi &\xrightarrow{x \rightarrow \infty} \frac{1}{x^2} \sum_{i=1}^N \rho_i^2 \\ B_\mu &\xrightarrow{x \rightarrow \infty} -\frac{2i\sigma_{\mu\nu} x_\nu}{x^2} \end{aligned}$$

This shows, surprisingly at first, that B_μ has the same asymptotic behavior as the BPST **instanton** (not anti-instanton) despite the fact that we are dealing with ASD solution². In any case, this means that B_μ approaches a pure gauge

$$\begin{aligned} B_\mu &\xrightarrow{x \rightarrow \infty} = g^{-1} \partial_\mu g \\ g &= \frac{x_0 + i\tau_i x_i}{r} \end{aligned}$$

Now define \bar{k} to be the winding number **as defined solely by the asymptotic behavior**. Then, obviously,

$$\bar{k}(B_\mu) = 1$$

$c = 1$ case: The case of $c = 1$ is more puzzling. In this case,

$$\varphi \xrightarrow{x \rightarrow \infty} 1 + \mathcal{O}(1/x^2) \quad \Rightarrow \quad B_\mu \xrightarrow{x \rightarrow \infty} \mathcal{O}(1/x^3)$$

² $F_{\mu\nu}$ is indeed still ASD.

and hence $\bar{k} = 0$.

What is happening is that **\bar{k} need not coincide with the true winding number k which is properly defined only for a regular solution.** To see this, we must study whether we can remove the singularity by a gauge transformation. We will do this one at a time.

First look at **the behavior around $x = a_1$.** One easily finds (for general c)

$$\varphi \xrightarrow{x \rightarrow a_1} \frac{\rho_1^2}{(x - a_1)^2} + \underbrace{\sum_{j=2}^N \frac{\rho_j^2}{(a_1 - a_j)^2}}_{c_1^2} + c^2 + \mathcal{O}\left(\frac{|x - a_1|}{a}\right)$$

where $a \equiv \min_{j \neq 1} |a_1 - a_j|$

For simplicity, let us define

$$\begin{aligned} c_1^2 &\equiv c^2 + \sum_{j=2}^N \frac{\rho_j^2}{(a_1 - a_j)^2} \\ \rho^2 &\equiv \frac{\rho_1^2}{c_1^2} \\ y &\equiv x - a_1 \end{aligned}$$

Then the behavior above takes the form

$$\varphi \xrightarrow{y \rightarrow 0} c_1^2 \left(1 + \frac{\rho^2}{y^2}\right) + \mathcal{O}(|y|/a)$$

From this one finds

$$\begin{aligned} B_\mu &\xrightarrow{y \rightarrow 0} \frac{-2i\rho^2\sigma_{\mu\nu}y_\nu}{y^2(y^2 + \rho^2)} + \mathcal{O}(y^2/a^2) \\ &= \frac{-2i\sigma_{\mu\nu}y_\nu}{y^2} + \frac{y^2}{y^2 + \rho^2} \left(\frac{-2i\sigma_{\mu\nu}y_\nu}{y^2}\right) + \mathcal{O}(y^2/a^2) \end{aligned}$$

This means that **the singular part of B_μ at $y = 0$ is a pure gauge** and can be removed by the **inverse** of the “large” gauge

transformation $g(y)$ given previously. (This procedure should be regarded as a mere technique of getting a regular solution.) The result of this procedure is

$$\begin{aligned}
B'_\mu &= gB_\mu g^{-1} + g\partial_\mu g^{-1} \\
&= \frac{y^2}{y^2 + \rho^2} g\partial_\mu g^{-1} + \mathcal{O}(y^2/a^2) \\
&= \frac{2i\bar{\sigma}_{\mu\nu}y_\nu}{y^2 + \rho^2} + \mathcal{O}(y^2/a^2) \quad \Leftarrow g\partial_\mu g^{-1} = \frac{2i\bar{\sigma}_{\mu\nu}y_\nu}{y^2}
\end{aligned}$$

Thus by this procedure, we indeed get around $y = 0$ a **regular anti-self-dual structure**. Now since we have performed a gauge transformation by g^{-1} , the winding number \bar{k} is now decreased by one unit. Thus **every time we remove the singularity by a gauge transformation we have $\Delta\bar{k} = -1$** . So, after removing N singularities, we get

$$\begin{aligned}
k &= \bar{k} = 1 - N = -(N - 1) && \text{for } c = 0 \\
k &= \bar{k} = 0 - N = -N && \text{for } c = 1
\end{aligned}$$

Exercise: Compute the winding number directly from the singular solution. (Hint: Utilize the fact that $\partial^2\partial^2 \ln \prod_{i=1}^N (x - a_i)^2 = 0$ for $x \neq a_i$.)