Chapter 2

Instantons in Quantum Mechanics

Before describing the instantons and their effects in non-abelian gauge theories, it is instructive to become familiar with the relevant ideas in the context of quantum mechanics.

2.1 Calculation of the ground state energy via path integral

We will be interested in the structure of the vacuum \emph{i.e.} the ground state of a quantum mechanical system. A powerful method is the path integral approach.

2.1.1 Formula for the transition amplitude

As is well-known, the property of the spectrum is reflected in the transition amplitude. Below we will review the method of extracting the information of the ground state.

Consider a particle in a potential $V(x)$. We will use the system of units with $\hbar = 1$, $m = 1$. Thus, the Lagrangian and
the Hamiltonian are
\[
L = \frac{1}{2} \dot{x}^2 - V(x)
\]
\[
H = \frac{1}{2} p^2 + V(x)
\]  

\[\text{(2.1)}\] 
\[\text{(2.2)}\]

\[\Box\] **Dimensional analysis and recovery of } \hbar \text{ and } \mathbf{m} : \]

In this system, we may take the length \( L \) to be the only fundamental scale. Let us analyze the dimension of various quantities in this scheme:

First, consider the momentum \( p \). Setting \( \hbar = 1 \) means

\[
\hbar = 1 = pL \implies p \sim \frac{1}{L}
\]

(2.3)

On the other hand, setting \( m = 1 \) means

\[
m = 1 \implies p \sim v \sim \frac{L}{T} \sim \frac{1}{L} \leftarrow \text{from (2.3)}
\]

\[
\therefore \quad T \sim L^2
\]

(2.4)

One can easily figure out the \( L \)-weight of other quantities. For example,

\[
E \sim \frac{mv^2}{L^2} \sim v^2 \sim \frac{1}{L^2}
\]

\[
\omega \sim \frac{1}{T} \sim \frac{1}{L^2}
\]

To recover \( \hbar \) and \( m \) from any expression, let us write quantities expressed in this system with a tilde.

Now the actual dimension of \( \hbar \) is

\[
\hbar \sim px \sim m \frac{L}{t} L \sim m \frac{L^2}{t} = m \tilde{t}
\]

From this we get

2.1-2
\[
\tilde{t} = \frac{\hbar}{m} t
\]

This is the only relation needed to recover the dependence on \(\hbar\) and \(m\).

**Examples:**

Consider first the case of the momentum. Since \(\tilde{m} = 1, \tilde{v} = L/\tilde{t}\), we get

\[
\tilde{p} = \tilde{m}\tilde{v} = \frac{L}{\tilde{t}} = \frac{mL}{\hbar t} = \frac{1}{\hbar} mv = \frac{p}{\hbar}
\]

As for the energy,

\[
\tilde{E} = \tilde{m}\tilde{v}^2 = \frac{L^2}{\tilde{t}^2} = \frac{m^2 L^2}{\hbar^2 t^2} = \frac{m}{\hbar^2} mv^2 = \frac{m}{\hbar^2} E
\]

Since we will omit tilde in the actual calculation, these relations can be implemented by the substitution rules such as

\[
p \to \frac{p}{\hbar}, \quad E \to \frac{m}{\hbar^2} E
\]

and so on.

\(\Box\) **Transition amplitude:**

Transition amplitude is given by

\[
T_{fi}(t_0) = \langle x_f | e^{-i t_0 H} | x_i \rangle = N \int D x e^{i S} = \sum_n e^{-i t_0 E_n} \langle x_f | n \rangle \langle n | x_i \rangle
\]  

(2.5)
where \( |n\rangle \) is the energy eigenstate \( H|n\rangle = E_n|n\rangle \).

We assume that the ground state is non-degenerate and its energy \( E_0 \) satisfies \( 0 < E_0 < E_n \). Then, the information of the ground state is obtained as follows:

- Make a Wick rotation to the Euclidean space by \( t = -i\tau \). (Rule: Demand \( e^{iS_M} = e^{-S_E} \), \( S_E > 0 \).)
- Take the limit \( \tau_0 \to \infty \). If we do not take this limit, then we may obtain the information of the entire spectrum, as we shall see later.

In the \( \tau_0 \to \infty \) limit, the sum over the intermediate states is dominated by the ground state:

\[
T_{fi}(\tau_0 \to \infty) \approx e^{-\tau_0 E_0} \psi_0(x_f) \psi_0^*(x_i) \quad (2.6)
\]

On the other hand, the actual path integral is given by

\[
T_{fi}(\tau_0) = N \int \mathcal{D}x(\tau) e^{-S_E} \quad (2.7)
\]

\[
S = S_E = \int_{-\tau_0/2}^{\tau_0/2} d\tau \left[ \frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \quad (2.8)
\]

This is well-defined if \( V(x) \) is bounded from below.

To compute this path integral, split the configuration \( x(\tau) \) into a classical solution \( X(\tau) \) and the quantum fluctuation \( \delta x(\tau) \).
around it:

\[ x(\tau) = X(\tau) + \delta x(\tau) \]  \hspace{1cm} (2.9) \\
\[ \delta x(\tau) = \sum_n c_n x_n(\tau) \]  \hspace{1cm} (2.10) \\
\[ X(\tau) = \text{A classical solution satisfying the B.C.} \]

\[ X(-\tau_0/2) = x_i, \quad X(\tau_0/2) = x_f \]

\( x_n(\tau) = \text{a complete orthonormal basis of functions in the given interval.} \) A convenient choice will be made later.

\[ \int_{-\tau_0/2}^{\tau_0/2} d\tau x_n(\tau)x_m(\tau) = \delta_{nm} \]  \hspace{1cm} (2.11) \\
\[ x_n(-\tau_0/2) = x_n(\tau_0/2) = 0 \]  \hspace{1cm} (2.12)

Then, up to the overall normalization, the path integral measure is given by

\[ \mathcal{D}x(\tau) = \prod_n \frac{dc_n}{\sqrt{2\pi}} \]  \hspace{1cm} (2.13)

**Classical Solution:**

The equation of motion is

\[ \frac{d^2x}{d\tau^2} = V'(x) \]  \hspace{1cm} (2.14)

where \( V'(x) \equiv \frac{dV}{dx} \)

Note that due to Euclideanization, the sign of the potential is effectively reversed.

Let us expand the action around this configuration. The term linear in \( \delta x \) vanishes by the equation of motion and we get

\[ S[X + \delta x] = S_0 + \frac{1}{2} \int_{-\tau_0/2}^{\tau_0/2} d\tau \delta x \hat{W} \delta x + \mathcal{O}(\delta x^3) \]  \hspace{1cm} (2.15) \\
\[ S_0 = S[X] \]

\[ \hat{W} = -\partial^2_x + V''(X) = \text{wave operator} \]  \hspace{1cm} (2.16)
As is usual, let us take \( x_n(\tau) \) to be the eigenfunctions of the hermitian operator \( \hat{W} \):

\[
\hat{W} x_n(\tau) = \epsilon_n x_n(\tau) \quad (\epsilon_n \sim L^{-4})
\]  

(2.17)

\[
\delta x = \sum c_n x_n
\]

(2.18)

Performing the \( \tau \)-integral using the orthonormality of \( x_n(\tau) \), and ignoring \( O(\delta x^3) \) terms (1-loop approximation), we get

\[
S = S_0 + \frac{1}{2} \sum_n \epsilon_n c_n^2
\]

(2.19)

Provided \( \epsilon_n > 0 \), the integral over \( c_n \) is a simple Gaussian integral:

\[
\int \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \epsilon_n c_n^2} = e^{-\frac{1}{2} \epsilon_n}
\]

(2.20)

Thus within this approximation, the Euclideanized transition amplitude becomes

\[
\langle x_f | e^{-\tau_0 H} | x_i \rangle = Ne^{-S_0} \prod_n \epsilon_n^{-\frac{1}{2}}
\]

(2.21)

Formally, since \( \prod_n \epsilon_n = \det \hat{W} \), we may write

\[
\prod_n \epsilon_n^{-\frac{1}{2}} = [\det (-\partial^2 + V''(X(\tau)))]^{-\frac{1}{2}}
\]

(2.22)

**Remarks**:

- The formula above depends only on \( \epsilon_n \) and does **not** require the knowledge of the behavior of the wave functions.
• As we shall see, the assumption $\epsilon_n > 0$ made above is not valid in general.
  There can be zero modes with $\epsilon_n = 0$. We will discuss in detail how to treat them later.

• The normalization factor $N$ has not been determined yet.

2.1.2 A simple application Harmonic oscillator

As the simplest example, consider the harmonic oscillator with the potential

$$V(x) = \frac{1}{2} \omega^2 x^2$$

In this case, the calculations can be performed exactly.

The equation of motion is

$$\frac{d^2 x}{d\tau^2} = \omega^2 x \quad (2.23)$$

We wish to expand around a classical solution with finite action.

If we take the simple boundary condition $x_i = x_f = 0$, the only such solution is the trivial solution $X(\tau) = 0$.

Since $V''(X = 0) = \omega^2$, the wave operator $\hat{W}$ is

$$\hat{W} = -\partial^2_\tau + \omega^2 \quad (2.24)$$

To compute $\det \hat{W}$, we need the eigenfunctions. The eigenvalue equation reads

$$\hat{W} x_n = \epsilon_n x_n \quad (2.25)$$

$$\Rightarrow \quad \ddot{x}_n + (\epsilon_n - \omega^2)x_n = 0 \quad (2.26)$$

$$x_n(\pm \tau_0/2) = 0 \quad (2.27)$$

The solution satisfying the B.C. at the left end, $x_n(-\tau_0/2) = 0$, is

$$x_n = A \sin \sqrt{\epsilon_n - \omega^2} \left( \tau + \frac{1}{2} \tau_0 \right) \quad (2.28)$$
Further imposing the condition at the right end, \( x_n(\tau_0/2) = 0 \), we get \( \sqrt{\epsilon_n - \omega^2\tau_0} = n\pi \). Since the LHS is non-negative, \( n \) runs over the range \( n \geq 0 \). However, we must remove the \( n = 0 \) case since then \( x_0 \) vanishes identically. Thus the eigenvalues are

\[
\epsilon_n = \left( \frac{n\pi}{\tau_0} \right)^2 + \omega^2, \quad n = 1, 2, \ldots \quad (2.29)
\]

From this knowledge we get

\[
N \prod_n \epsilon_n^{-1/2} = N \prod_{n=1}^{\infty} \left( \frac{n^2\pi^2}{\tau_0^2} \right)^{-1/2} \cdot \left( \prod_{n=1}^{\infty} \left( 1 + \frac{\omega^2\tau_0^2}{n^2\pi^2} \right) \right)^{-1/2}
\]

The factor \( A \) must coincide with the contribution of a free particle \( i.e. \) when \( \omega = 0 \). This means

\[
A = \langle x_f = 0 | e^{-\frac{1}{2}p^2\tau_0} | x_i = 0 \rangle = \int \frac{dp}{2\pi} \langle x_f = 0 | e^{-\frac{1}{2}p^2\tau_0} | p \rangle \langle p | x_i = 0 \rangle = \int \frac{dp}{2\pi} e^{-\frac{1}{2}p^2\tau_0} = \frac{1}{\sqrt{2\pi \tau_0}} \quad (2.30)
\]

(If we wish, we may determine \( N \) by computing the infinite product using \( \zeta \)-function regularization. But this is not needed.)

On the other hand the factor \( B \) can be computed using the following well-known formula:

2.1-8
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{y^2}{n^2} \right) = \frac{\sinh \pi y}{\pi y} \quad (2.31)
\]

This formula can be understood in the following way: Consider the function \( \pi \cot \pi z \). This has 1st order poles at \( z = 0, \pm 1, \pm 2, \ldots \) with residue all equal to 1. Thus, we must have

\[
\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{z+n} \right) + f(z) \quad (2.32)
\]

where \( f(z) \) is regular on the complex plane. Moreover, one can show that \( f(0) = 0 \). Such an analytic function must identically vanish.

Now it is easy to show that

\[
\frac{d}{dz} \ln \left( \frac{\sin \pi z}{\pi z} \right) = \pi \cot \pi z - \frac{1}{z} \quad (2.33)
\]

Therefore, by term-by-term integration in the interval \( 0 \leq z \leq 1 \) (justified since the sum is uniformly convergent in this interval) one gets

\[
\ln \left( \frac{\sin \pi z}{\pi z} \right) = \sum_{n=1}^{\infty} \left[ \ln \left( \frac{z-n}{-n} \right) + \ln \left( \frac{z+n}{n} \right) \right]
\]

\[
= \sum_{n=1}^{\infty} \ln \left( 1 - \frac{z^2}{n^2} \right) \quad (2.34)
\]

\[
\therefore \quad \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \quad (2.35)
\]

Setting \( z = iy \) we get the formula above.
Now set $\pi y = \omega \tau_0$ and combine with the factor of $A$. We get

$$\langle x_f = 0 | e^{-\tau_0 H} | x_i = 0 \rangle = \frac{1}{\sqrt{2\pi \tau_0}} \left( \frac{\sinh \omega \tau_0}{\omega \tau_0} \right)^{-1/2} = \left( \frac{\omega}{\pi} \right)^{1/2} (2 \sinh \omega \tau_0)^{-1/2}$$

$$= \left( \frac{\omega}{\pi} \right)^{1/2} \left( e^{\omega \tau_0} - e^{-\omega \tau_0} \right)^{-1/2}$$

$$= \left( \frac{\omega}{\pi} \right)^{1/2} e^{-\omega \tau_0/2} \left( 1 - e^{-2\omega \tau_0} \right)^{-1/2}$$

$$= \left( \frac{\omega}{\pi} \right)^{1/2} e^{-\omega \tau_0/2} \left( 1 + \frac{1}{2} e^{-2\omega \tau_0} + \cdots \right) \quad (2.36)$$

Since we have not taken the limit $\tau_0 \to \infty$, this must equal $\sum_n e^{-\tau_0 E_n} | \psi_n(0) |^2$.

Since $\psi_n(0) = 0$ for odd $n$ (due to our choice of the boundary condition), we find the energy spectrum for all the even energy levels

$$E_{2n} = \frac{\omega}{2} + 2n\omega \quad (2.37)$$

This agrees exactly with the (part of) the spectrum of a harmonic oscillator.

Especially, for the ground state, we get

$$E_0 = \frac{\omega}{2}, \quad \psi_0(0) = \left( \frac{\omega}{\pi} \right)^{1/4} \quad (2.38)$$

2.1-10
2.2 System in a double-well potential: Instanton calculation

Now we apply the method above to the system of a particle in a double well potential, the simplest system for which the instanton method is applicable.

We take the potential with two symmetric wells with the bottoms located at $x = \pm a$. Around each bottom, it is assumed to have the expansion

$$V(x) = \frac{\omega^2}{2}(x - a)^2 + O((x - a)^3) \quad (2.39)$$

- It is well-known that the lowest energy states localized at two bottoms are not the true ground states.
- They get mixed by the quantum mechanical tunneling effect and the energy becomes lower for the symmetric combination.
- Our purpose is to understand this phenomenon from the path-integral point of view.

Although no further information is needed for WKB approximation, to be performed later, we shall be more specific and fix our potential to be

$$V(x) = \lambda(x^2 - a^2)^2 \quad (2.40)$$

\[\begin{array}{c}
\hline
\text{Diagram showing a double well potential with bottoms at } -a \text{ and } a. \\
\hline
\end{array}\]

2.2-11
An advantage of this form is that the calculation of the relevant determinant can be performed **exactly**, as we shall discuss below.

We list some derivatives of $V$:

\[
V'(x) = 4\lambda x(x^2 - a^2), \quad V'(\pm a) = 0 \\
V''(x) = 4\lambda(3x^2 - a^2), \quad V''(\pm a) = \omega^2 = 8\lambda a^2 \\
V'''(x) = 24\lambda x, \quad V'''(\pm a) = \pm 24\lambda a
\]

### 2.2.1 Classical Solution

As before, we first look for classical solutions. The Euclideanized equation of motion is

\[
\frac{d^2 X}{d\tau^2} = V'(X) \tag{2.41} \\
V(X) = \lambda(X^2 - a^2)^2 \tag{2.42}
\]

This is readily solved by multiplying both sides by $\dot{X}$:

\[
\dot{X}\ddot{X} = \frac{1}{2}\frac{d\tau}{d\tau}(\dot{X})^2 = \partial_\tau V(X) \\
\therefore \quad \frac{1}{2}(\dot{X})^2 = V(X) + c, \quad c \geq 0 \tag{2.43}
\]

As we will take the limit $\tau_0 \rightarrow \infty$ later, we want a solution with **finite action** in this limit. From the relation above,

\[
S[X] = \int_{-\infty}^{\infty} d\tau(\dot{X}^2 - c) \tag{2.44}
\]

For this to be finite, we need $\dot{X} = \pm \sqrt{c}$ for large $|\tau|$. Then from the equation of motion (2.41), we must have $0 = V'(X)$ in this region. Namely the particles must be at rest at the top of the **inverted** potential $X = \pm a$. This requires $c = 0$.

Thus there are only 3 types of solutions:
• Particle stays at the top:

\[
\begin{align*}
    X(\tau) &= a \quad \text{for all } \tau \\
    X(\tau) &= -a \quad \text{for all } \tau
\end{align*}
\]

The contributions from small fluctuations around this solution is essentially the same as for the harmonic oscillator case and has already been computed.

• Particle leaves \( x = -a \) at \( -\tau_0/2 \) and reaches \( x = a \) at \( \tau_0/2 \):

Such a solution is called an (anti-)instanton.

• Particle oscillates between the summits. We will consider this case later.

Explicit solution: Putting in the explicit form of \( V(x) \), we have

\[
\begin{align*}
    \dot{X} &= \pm \sqrt{2\lambda} (a^2 - X^2), \quad X^2 \leq a^2 \\
    \frac{dX}{a^2 - X^2} &= \pm \sqrt{2\lambda} (\tau - \tau_c)
\end{align*}
\]

This can be easily integrated. Imposing the boundary condition \( X(\infty) = \pm a \) we get the following solitonic solution:

\[
    X = \pm a \tanh \sqrt{2\lambda} a (\tau - \tau_c)
\]  \hspace{1cm} (2.45)

Since \( 8\lambda a^2 = \omega^2 \), this can be written as

\[
    X_\pm = \pm a \tanh \left( \frac{\omega}{2} (\tau - \tau_c) \right) \xrightarrow{\tau \to \infty} \pm a \quad (2.46)
\]

\( +, - \) refer to instanton and anti-instanton respectively. They are depicted as

2.2-13
Let us compute the action for such a solution. Using the relation $V(X) = \frac{1}{2} \dot{X}^2$, we get

$$S_0 = S[X] = \int_{-\infty}^{\infty} d\tau \dot{X}^2$$

$$= \frac{a^2 \omega^2}{4} \int_{-\infty}^{\infty} d\tau \frac{1}{\cosh^4 \frac{\omega}{2}(\tau - \tau_c)} = \frac{\omega^3}{16 \lambda} \int_{-\infty}^{\infty} du \frac{1}{\cosh^4 u}$$

$$u \equiv \frac{\omega}{2}(\tau - \tau_c)$$

The integral is easily done by putting $t = \tanh u$ and gives 4/3. Thus,

$$S_0 = \frac{\omega^3}{12 \lambda} \quad (2.47)$$

### 2.2.2 Calculation of the prefactor determinant

The transition amplitude for the process $x = -a$ to $x = a$ takes the form

$$\langle a | e^{-\tau_0 H} | -a \rangle = N \det \left( -\partial_{\tau}^2 + V''(X) \right)^{-1/2} e^{-S_0}$$

2.2-14
To compute the determinant, we take the one for the harmonic oscillator as the normalization. Hence we write
\[ N \det \left( -\partial^2_\tau + V''(X) \right)^{-1/2} = N \det \left( -\partial^2_\tau + \omega^2 \right)^{-1/2} \times \left\{ \frac{\det \left( -\partial^2_\tau + V''(X) \right)}{\det \left( -\partial^2_\tau + \omega^2 \right)} \right\}^{-1/2} \]
Since we have already computed the first factor, what we want is the ratio in the 2nd factor. For this purpose, we must solve the eigenvalue problem.

By a simple calculation,
\[ V''(X) = \omega^2 \left( 1 - \frac{3}{2} \cosh^2 \frac{\omega}{2} (\tau - \tau_c) \right) \]  
Thus, the eigenvalue equation needed for the calculation of the determinant becomes
\[ -\frac{d^2 x_n(\tau)}{d\tau^2} + V''(X(\tau)) x_n(\tau) = \epsilon_n x_n(\tau) \]  
This is formally identical to the Schrödinger equation in the potential \( U(\tau) = V''(X(\tau)) \) (with \( \tau \) playing the role of \( x \)).

From the form of this potential, it is clear that there are discrete as well as continuous spectra.
General solution of the Schrödinger equation:

The explicit form of the equation is

\[ \partial^2_{\tau}x_n - (\omega^2 - \epsilon_n)x_n + \frac{3\omega^2}{2\cosh^2 u}x_n = 0 \quad (2.50) \]

where \[ u = \frac{\omega}{2}(\tau - \tau_c) \]

As we have already mentioned, this equation turns out to be exactly solvable.

We will simplify the equation in steps. First make a change of variables

\[ \xi \equiv \frac{X_+}{a} = \tanh u, \quad X_+ = \text{instanton sol} \]

Then,

\[ \dot{\xi} = \frac{\omega}{2}(1 - \xi^2) \]

\[ \therefore \partial_\tau = \frac{\omega}{2}(1 - \xi^2) \partial_\xi \]

\[ \frac{1}{\cosh^2 u} = 1 - \xi^2 \]

In terms of \( \xi \), the equation becomes

\[ \frac{d}{d\xi}(1 - \xi^2)\frac{d}{d\xi}x_n + \left( a + \frac{b}{1 - \xi^2} \right) x_n = 0 \quad (2.51) \]

where \[ a = 6, \quad b = \frac{4(\epsilon_n - \omega^2)}{\omega^2} \quad (2.52) \]

This type of equation can be solved in terms of hypergeometric function as follows. Let \( c \) be a constant and set (we omit the subscript \( n \))

\[ x = (1 - \xi^2)^c \chi \]

(\( c \) will be chosen appropriately later.) Then, after a simple algebra, we get

\[ (1 - \xi^2)\frac{d^2\chi}{d\xi^2} - (4c + 2)\xi \frac{d\chi}{d\xi} + \left( a - 2c - 4c^2 + \frac{b + 4c^2}{1 - \xi^2} \right) \chi = 0 \]

2.2-16
Further make a change of variable of the form

\[ z = \frac{1}{2} (1 - \xi) \]

\[ \therefore 1 - z = \frac{1}{2} (1 + \xi), \quad 1 - \xi^2 = 4z(1 - z) \]

Correspondence of the asymptotic values of variables is

\[
\begin{align*}
\tau \to \infty & \iff \xi \to 1 & \iff z \to 0 \\
\tau \to -\infty & \iff \xi \to -1 & \iff z \to 1
\end{align*}
\]

Then the equation is further simplified to

\[
0 = z(1-z)\frac{d^2\chi}{dz^2} + (2c + 1 - (4c + 2)z) \frac{d\chi}{dz} + \left(a - 2c - 4c^2 + \frac{b + 4c^2}{4z(1-z)}\right)\chi
\]

Now if we choose \( c \) such that

\[ b + 4c^2 = 0, \]

this becomes the familiar hypergeometric equation

\[
z(1-z)\frac{d^2\chi}{dz^2} + (\gamma - (\alpha + \beta + 1)z) \frac{d\chi}{dz} - \alpha\beta\chi = 0
\]

with

\[
\gamma = 2c + 1, \quad \alpha + \beta = 4c + 1, \quad \alpha\beta = 4c^2 + 2c - a
\]

The solution is the hypergeometric function \( F(\alpha, \beta, \gamma; z) \), which in the vicinity of \( z = 0 \) has the expansion

\[
F(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha\beta z}{\gamma 1!} + \frac{\alpha(\alpha + 1)\beta(\beta + 1) z^2}{\gamma(\gamma + 1) 2!} + \cdots
\]

2.2-17
In our case with $a = 6$, the parameters simplify to\(^1\)

\[
\alpha = 2c - 2, \quad \beta = 2c + 3, \quad \gamma = 2c + 1
\]

**Continuous spectrum:**

For $\epsilon \geq \omega^2$, if we do not impose any asymptotic conditions, the spectrum is continuous, which can be parametrized by the real positive momentum

\[
p \equiv \sqrt{\epsilon - \omega^2}, \quad \text{or} \quad k \equiv \frac{p}{\omega}
\]

(2.53)

Since there is no barrier for $\epsilon > \omega^2$, we expect that the particle does not get reflected at all as it travels from $\tau = -\infty$ to $\tau = \infty$. This means that all the scattering dynamics is contained in the knowledge of the **phase shift** $\delta_p$ defined here as (set $\tau_c = 0$)

\[
x_p(\tau \to \infty) = e^{ip\tau} \\
x_p(\tau \to -\infty) = e^{ip\tau + i\delta_p}
\]

As it will be shown, to compute the contribution to the determinant, the knowledge of $\delta_p$ will be enough.

Now for $\epsilon > \omega^2$, we have (recall $b = \frac{4(\epsilon_n - \omega^2)}{\omega^2}$) $b = 4k^2 = -4c^2$, so we get $c^2 = -k^2$. The solution which asymptotes to $e^{ip\tau}$ as $\tau \to \infty$ (i.e. $z \to 0$) is, choosing $c = -ik$,

\[
x = (1 - \xi^2)^{-ik} F(\alpha, \beta, \gamma; z)
\]

In fact, near $\tau \to \infty$, (using $u = \frac{1}{2}\omega\tau$ and hence $2uk = \omega\tau k = p\tau$)

\[
(1 - \xi^2)^{-ik} = (\cosh^2 u)^{ik} \simeq (4e^{-2u})^{-ik} = 4^{-ik}e^{ip\tau} \quad (2.54)
\]

\(^1\)The role of $\alpha$ and $\beta$ can of course be interchanged.
**Phase shift:** To find the phase shift, we must *analytically continue* this solution to the one valid around $z = 1$. As for the front factor, near $\tau = -\infty$, 

$$
(1 - \xi^2)^{-ik} \simeq (4e^{+2u})^{-ik} = 4^{-ik}e^{-ip\tau}
$$

(2.55)

Thus, this alone represents the *reflected wave*. On the other hand, the hypergeometric function must be rewritten as

$$
F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta + 1 - \gamma; 1 - z)
$$

$$
F_1(1 - z) = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{\gamma - \alpha - \beta}
$$

$$
\times F(\gamma - \alpha, \gamma - \beta, \gamma + 1 - \alpha - \beta; 1 - z)
$$

where

$$
\alpha = -2ik - 2, \quad \beta = -2ik + 3, \quad \gamma = -2ik + 1
$$

It is easy to see that $F_1(1 - z)$ vanishes due to the denominator factor $\Gamma(\gamma - \beta) = \Gamma(-2)$ which diverges. This shows that indeed there is no reflected wave.

The remaining part $(1 - \xi^2)^{-ik}F_2(1 - z)$ becomes

$$
(1 - \xi^2)^{-ik}F_2(1 - z) = \frac{\Gamma(-2ik + 1)\Gamma(-2ik)}{\Gamma(-2ik + 3)\Gamma(-2ik - 2)} (1 - z)^{2ik}
$$

$$
\times (1 - \xi^2)^{-ik}(1 + O(1 - z))
$$

$$
= \frac{(1 + 2ik)(1 + ik)}{(1 - 2ik)(1 - ik)} (1 - \xi^2)^{ik}4^{-2ik}(1 + O(1 - z))
$$

$$
\approx \frac{(1 + 2ik)(1 + ik)}{(1 - 2ik)(1 - ik)} 4^{-ik}e^{i\pi \tau}
$$

(2.56)

Comparing with (2.54), we can read off the phase shift as
\[ e^{i\delta_p} = \frac{(1 + 2ik)(1 + ik)}{(1 - 2ik)(1 - ik)} \quad (2.57) \]

**Discretization of the spectrum** To compute the contribution to the determinant, we must regularize the continuous spectrum. The simplest way is to put the system in a box of interval \(-\tau_0/2 < \tau < \tau_0/2\).

As the boundary condition, we take
\[ x(\tau_0/2) = x(-\tau_0/2) = 0 \]

Because of this boundary condition, **there will be a reflected wave** (which vanishes as \(\tau_0 \to \infty\)) present and we must consider the general solution. Since \(x_p(-\tau)\) is obviously a solution independent of \(x_p(\tau)\), such a solution is
\[ x = Ax_p(\tau) + Bx_p(-\tau) \]

Applying the boundary condition above, we get
\[ Ax_p(\tau_0/2) + Bx_p(-\tau_0/2) = 0 \]
\[ Ax_p(-\tau_0/2) + Bx_p(\tau_0/2) = 0 \]

Non-trivial solution exists iff \(A = \pm B\). This means
\[ \frac{x_p(\tau_0/2)}{x_p(-\tau_0/2)} = \pm 1 \]

Using the asymptotic form of the solution, the LHS translates into \(e^{ip\tau_0 - i\delta_p} = \pm 1\). Hence the solutions are
\[ \tilde{p}_n = \frac{\pi n + \delta_p}{\tau_0}, \quad n = 0, 1, 2, \ldots \]

where we denote by \( \tilde{p}_n \) the (by definition positive) parameter \( p \) satisfying the above condition.

**Contribution to the determinant ratio:** We now compute the contribution of these modes to the ratio of the determinant. Recall that for the harmonic oscillator case, the spectrum is

\[ p_n = \frac{\pi n}{\tau_0} \]

It is clear that in the limit \( \tau_0 \to \infty \), contribution from a finite number of eigenvalues with \( n \sim O(1) \) cancel against the contribution from the harmonic oscillator states as they both become \( \omega^2 \). Thus, the ratio to be computed can be taken as (by shifting a few levels and recalling \( p = \sqrt{\epsilon - \omega^2}, \epsilon = \omega^2 + p^2 \))

\[ R \equiv \prod_{n=1}^{\infty} \frac{\omega^2 + \tilde{p}_n^2}{\omega^2 + p_n^2} \]

As \( \tau_0 \to \infty \), the difference \( \Delta p_n \equiv \tilde{p}_n - p_n = \delta_p / \tau_0 \to 0 \). Thus, we can expand in powers of \( \Delta p_n \) to the first order. Hence,

\[ R = \exp \left( \sum \ln \frac{\omega^2 + \tilde{p}_n^2}{\omega^2 + p_n^2} \right) \approx \exp \left( \sum \frac{2p_n \Delta p_n}{\omega^2 + p_n^2} \right) \]

At the same time the interval \( \pi/\tau_0 \) goes to zero and we may convert this into the integral by noting

\[ \Delta p_n = \frac{\delta_p}{\tau_0} = \frac{\delta_p}{\pi} \frac{\pi}{\tau_0} = \frac{\delta_p}{\pi} (p_{n+1} - p_n) = \frac{\delta_p}{\pi} \Delta p_n \quad (2.58) \]

2.2-21
So

\[ R \simeq \exp \left( \frac{1}{\pi} \int_{0}^{\infty} \delta_p \frac{2p}{\omega^2 + p^2} dp \right) \]

\[ = \exp \left( \frac{1}{\pi} \int_{0}^{\infty} \delta_p \frac{d}{dp} \ln \left( 1 + \frac{p^2}{\omega^2} \right) dp \right) \]

\[ = \exp \left( -\frac{1}{\pi} \int_{0}^{\infty} dk \frac{d\delta_k}{dk} \ln \left( 1 + k^2 \right) \right) \]

where in the last line we used the integration by parts. From the explicit expression (2.57) for the phase shift, we easily get

\[ \frac{d\delta_k}{dk} = \frac{2}{1 + k^2} + \frac{4}{1 + 4k^2} \]

Now use the formula

\[ \int_{0}^{\infty} dk \frac{\ln(1 + k^2)}{1 + a^2k^2} = \frac{\pi}{a} \ln \left( 1 + \frac{1}{a} \right) \]

Then we easily get

\[
R = e^{-\ln 9} = \frac{1}{9} \quad (2.59)
\]

Remarkably, we have been able to compute the ratio of the determinant exactly!

**Discrete spectrum:**

Now we consider the discrete part of the spectrum in the case \( \omega^2 - \epsilon_n > 0 \). Thus, we should set

\[ c = k \equiv \frac{\sqrt{\omega^2 - \epsilon_n}}{\omega} > 0 \]
The solution of the Schrödinger equation which is finite in the vicinity of $z = 0$ (i.e. $\tau \to \infty$) is

$$x_n = (1 - \xi^2)^k F(\alpha, \beta, \gamma; z)$$

(2.60)

$$F(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha \beta \gamma \tau}{\gamma \gamma + 1} + \frac{\alpha (\alpha + 1) (\beta + 1)}{\gamma (\gamma + 1) 2!} + \cdots$$

(2.61)

with

$$\alpha = 2k - 2, \quad \beta = 2k + 3, \quad \gamma = 2k + 1.$$ In order for this solution to be finite at $z = 1$ (i.e. as $\tau \to -\infty$), the series must terminate at finite terms. Since $\beta, \gamma, k > 0$ and $\alpha = 2k - 2 > -2$, it can occur only when

$$\alpha = 2k - 2 = -n, \quad n = 0, 1$$

(2.62)

$$\Rightarrow \quad k = \frac{1}{2}$$

Recalling the definition of $k$, this means that there are two discrete energy levels (bound states):

$$\epsilon_0 = 0, \quad \epsilon_1 = \frac{3}{4} \omega^2$$

(2.63)

The corresponding wave functions can be obtained using the form of the (truncated) hypergeometric function:

$$x_0(\tau) \propto \frac{1}{\cosh^2 \frac{\omega}{2} (\tau - \tau_c)} \overline{\tau_\to \infty} e^{-\omega (\tau - \tau_c)}$$

(2.64)

$$x_1(\tau) \propto \frac{\sinh \frac{\omega}{2} (\tau - \tau_c)}{\cosh^2 \frac{\omega}{2} (\tau - \tau_c)} \overline{\tau_\to \infty} e^{-\frac{\omega}{2} (\tau - \tau_c)}$$

(2.65)
2.2.3 Treatment of the zero mode

Of the two discrete modes found above, one corresponds exactly to the zero eigenvalue. Inclusion of such a mode in the determinant makes it vanish. Below, we discuss the meaning of the zero mode and how to treat it.

□ Normalization of the zero mode:

For later convenience, let us normalize the zero mode:

\[ x_0(\tau) = \frac{C}{\cosh^2 \frac{\omega}{2}(\tau - \tau_c)} \]

\[ 1 = \int_{-\infty}^{\infty} \frac{C^2}{\cosh \frac{\omega}{2}(\tau - \tau_c)} = C^2 \frac{8}{3\omega} \]

\[ \therefore x_0(\tau) = \sqrt{\frac{3\omega}{8}} \frac{1}{\cosh^2 \frac{\omega}{2}(\tau - \tau_c)} \] (2.66)

□ Zero mode and invariance of the theory:

The existence of the zero mode is actually due to the underlying translation invariance of the theory.

Of course, each instanton solution, with a specific \( \tau_c \) describing the position of its center, spontaneously breaks translation invariance. However the fact that its action does not depend on \( \tau_c \) is a reflection of translation invariance. For infinitesimal change of the central position, we have

\[ S[X(\tau, \tau_c + \delta \tau_c)] - S[X(\tau, \tau_c)] = 0 \] (2.67)

Expanding this with respect to \( \delta \tau_c \) we get

\[ 0 = \int d\tau \frac{\delta S}{\delta X(\tau)} \delta \tau_c \frac{\partial X}{\partial \tau_c}(\tau) \]

\[ + \frac{1}{2} \int d\tau d\tau' \delta \tau_c \frac{\partial X}{\partial \tau_c}(\tau') \frac{\delta^2 S}{\delta X(\tau) \delta X(\tau')} \delta \tau_c \frac{\partial X}{\partial \tau_c}(\tau) \]

\[ + \mathcal{O}(\tau_c^3) \]
The first line vanishes due to the equation of motion. As for the second line, note
\[
\frac{\delta^2 S}{\delta X(\tau) \delta X(\tau')} = \left[ -\partial_{\tau}^2 + V''(X(\tau)) \right] \delta(\tau - \tau') \tag{2.68}
\]
This shows that \( \partial X(\tau, \tau_c)/\partial \tau_c \) is the zero mode of the differential operator \( \hat{W} \).

Let us compute the normalized solution. Since \( \partial X(\tau, \tau_c)/\partial \tau_c = -\partial X(\tau, \tau_c)/\partial \tau \), we may write
\[
x_0(\tau) = C \frac{\partial X}{\partial \tau}
\]
Normalization condition reads
\[
1 = \int_{-\infty}^{\infty} d\tau (x_0(\tau))^2 = C^2 \int_{-\infty}^{\infty} d\tau \left( \frac{\partial X}{\partial \tau} \right)^2 = C^2 S_0
\]
\[
\therefore \quad C = S_0^{-\frac{1}{2}} \tag{2.69}
\]
Therefore the normalized solution can be written as
\[
x_0(\tau) = S_0^{-\frac{1}{2}} \frac{\partial X}{\partial \tau} \tag{2.70}
\]
Using the calculation performed previously, the RHS is
\[
S_0 = \frac{\omega^3}{12\lambda}
\]
\[
\frac{\partial X}{\partial \tau} = \frac{a\omega}{2} \frac{1}{\cosh^2 \frac{\omega}{2} \tau} = \frac{\omega}{\sqrt{8\lambda}} \frac{1}{2 \cosh^2 \frac{\omega}{2} \tau}
\]
\[
\therefore \quad x_0(\tau) = \sqrt{\frac{12\lambda}{\omega^3}} \sqrt{\frac{\omega^4}{32\lambda}} \frac{1}{\cosh^2 \frac{\omega}{2} \tau}
\]
\[
= \sqrt{3\omega} \frac{1}{8} \cosh^2 \frac{\omega}{2} \tau \tag{2.71}
\]
This agrees with the explicit result obtained before, without solving the Schrödinger equation!

□ **Integration over the zero mode: collective coordinate:**

Previously, we have performed the formal Gaussian integration over $c_n$ defined by

$$x(\tau) = \sum_n c_n x_n(\tau)$$

and obtained $(\det \hat{W})^{-1/2}$. However, for the zero mode, the integration is simply $\int dc_0$ without the Gaussian suppression and it obviously diverges, corresponding to the vanishing of the determinant. Thus, we must actually treat the zero-mode integration separately.

If we vary $c_0$ in (2.72), $x(\tau)$ changes by

$$\Delta x(\tau) = \Delta c_0 x_0(\tau)$$

which is proportional to $x_0(\tau)$. On the other hand, the variation of the position $\tau_c$ induces the change

$$\Delta x(\tau) = \Delta X(\tau) = \frac{dX}{d\tau_c} \Delta \tau_c = - \frac{dX}{d\tau} \Delta \tau_c = - \sqrt{S_0} x_0(\tau) \Delta \tau_c$$

Comparing, we see that these two types of variations are actually the same with the non-trivial Jacobian factor given by

$$J = \left| \frac{dc_0}{d\tau_c} \right| = \sqrt{S_0} \quad (2.73)$$

In this way, we find that the zero-mode integration should be replaced by the one over $\tau_c$. In other words, $\tau_c$ is promoted to a dynamical variable over which to integrate.
Such a variable is called a collective coordinate. Integration over $\tau_c$ restores translation invariance.

2.2.4 Complete one instanton contribution

Having understood the treatment of the zero mode, the only remaining contribution to the ratio of the determinant is from the level with $\epsilon = 3/4\omega^2$. In the limit $\tau_0 \to \infty$, this clearly contributes $3/4$ to the determinant ratio.

Combining all the results obtained, we get the complete one-instanton contribution for large $\tau_0$ as

$$
\langle -a | e^{-H \tau_0} | a \rangle_{\text{one-inst}} = \frac{1}{\sqrt{2\pi \tau_0}} \left( \frac{\sinh \omega \tau_0}{\omega \tau_0} \right)^{-1/2} \times \left( \frac{1}{\omega^2} \times \frac{3}{4} \times \frac{1}{9} \right)^{-1/2} e^{-S_0} \sqrt{S_0} \frac{d\tau_c}{\sqrt{2\pi}}
$$

(The factor of $1/\sqrt{2\pi}$ comes from the fact in trading $\epsilon_0$ for the harmonic oscillator with the zero mode integration we must take care of such a factor in $\int dc_0 e^{-\frac{1}{2}c_0^2} = \sqrt{\frac{2\pi}{\epsilon_0}}$ altogether.) Simplifying, we get

$$
\langle -a | e^{-H \tau_0} | a \rangle_{\text{one-inst}} = F_{HO} \cdot \rho \cdot d\tau_c \quad (2.74)
$$

$$
F_{HO} = \sqrt{\frac{\omega}{\pi}} e^{-\omega \tau_0/2} \quad (2.75)
$$

$$
\rho = \sqrt{\frac{6}{\pi}} \sqrt{S_0} e^{-S_0} = \text{indep. of } \tau_0 \quad (2.76)
$$

The $F_{HO}$ represents the harmonic oscillator-like contribution while $\rho$ can be regarded as a kind of density of an instanton.

2.3-27
2.3 Dilute Gas of Instantons and Anti-instantons

When the time interval $\tau_0$ is very large, we must actually take into account the possibility that the particle oscillates between the two tops of the inverted potential in such a large interval. This is depicted in the following figure:

Such a path consists of alternating instanton-anti-instanton (IA) configurations and if their separation $|\tau_i - \tau_j|$ is much bigger than the characteristic time scale $1/\omega$ of the system, it should constitute an approximate classical solution.

If there are $n$ such IA’s, the action should be $\sim nS_0$, where $S_0$ is the one-instanton action. Although their contribution is suppressed by $e^{-nS_0}$, they sum up to give a very important contribution, as we shall see.

□ Contribution of $n$ IA’s:

The contribution to the transition amplitude from $n$-IA configuration takes the form

$$A_n = F_{HO} \cdot \rho^n \cdot I_n \quad (2.77)$$

where

$$I_n = \int_{-\tau_0/2}^{\tau_0/2} \omega d\tau_n \cdots \int_{-\tau_0/2}^{\tau_3} \omega d\tau_2 \int_{-\tau_0/2}^{\tau_2} \omega d\tau_1 \quad (2.78)$$

$$\left(-\frac{\tau_0}{2} \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \frac{\tau}{2}\right) \quad (2.79)$$
To compute this integral, first define

\[ x_i \equiv \omega (\tau_i + (\tau_0/2)) \]
\[ x \equiv \omega (\tau + (\tau_0/2)) \]

Then, the integral can be written as \( I_n = J_n(x = \omega \tau_0) \) where

\[ J_n(x) = \int_0^x dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \]

Since

\[ \frac{dJ_n(x)}{dx} = J_{n-1}(x) \]

we have

\[ \frac{d^{n-1}J_n}{dx^{n-1}} = J_1(x) = x. \]

But since \( J_k(0) = 0 \) for all \( k \), we can immediately integrate this equation to get

\[ J_n(x) = \frac{x^n}{n!} \]

\[ \therefore \quad I_n = \frac{(\omega \tau_0)^n}{n!} \] (2.80)

\[ \Box \] Transition amplitudes and the energy spectrum:

Although it is sufficient to consider the amplitude
(a) \( \langle a | e^{-H \tau_0} | a \rangle \) for the purpose of computing the energy levels, it is instructive to consider
(b) \( \langle -a | e^{-H \tau_0} | a \rangle \) as well.

- For (a), the number of IA is even
- For (b), the number of IA is odd
Thus

\[
\langle a | e^{-H\tau_0} | a \rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \sum_{n=0,2,...} \frac{1}{n!} (\omega\tau_0\rho)^n
\]

\[
= \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \cosh \omega\tau_0\rho
\]

\[
= \sum_n e^{-E_n\tau_0} |\langle a | n \rangle|^2
\]

(2.81)

\[
\langle -a | e^{-H\tau_0} | a \rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \sum_{n=1,3,...} \frac{1}{n!} (\omega\tau_0\rho)^n
\]

\[
= \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_0/2} \sinh \omega\tau_0\rho
\]

\[
= \sum_n e^{-E_n\tau_0} \langle -a | n \rangle \langle n | a \rangle
\]

(2.82)

Note that \( e^{-E\tau_0} \) type expression arises only after the infinite sum.

From these expressions, one easily gets the following information:

\[
E_0 = \frac{\omega}{2} - \frac{\omega}{2} \sqrt{\frac{2\omega^3}{\pi \lambda}} e^{-\omega^3/12\lambda} \quad (2.83)
\]

\[
E_1 = \frac{\omega}{2} + \frac{\omega}{2} \sqrt{\frac{2\omega^3}{\pi \lambda}} e^{-\omega^3/12\lambda} \quad (2.84)
\]

\[
\langle a | 0 \rangle = \langle -a | 0 \rangle = \left( \frac{\omega}{4\pi} \right)^{1/4} \quad (2.85)
\]

We see that

- The energy is split by non-perturbative tunneling contribution.
- The ground state is indeed symmetric under reflection.

\[ \square \text{ Justification of the IA Gas:} \]

2.3-30
We must check that at least in some regime the dilute gas approximation employed above is valid. The characteristic separation between IA’s is given by \( \tau_0/n_c \), where \( n_c \) is the characteristic value of \( n \) contributing to the series \( \sum_n x^n/n! \). This would occur when \( x^n/n! \) regarded as a function of \( n \) is stationary. Using the Stirling’s formula,

\[
\frac{\partial}{\partial n} x^n/n! = \frac{\partial}{\partial n} e^{n \ln x - n \ln n} \simeq (\ln x - \ln n)e^{n \ln x - n \ln n} = 0
\]

\[\therefore \quad n_c \sim x = \omega \tau_0 \rho\]

\[\therefore \quad \frac{\tau_0}{n_c} \sim \frac{1}{\omega \rho} \gg \frac{1}{\omega} \quad \text{for small } \rho\]

This is certainly achieved for small enough coupling \( \lambda \).