Constrained Dynamical Systems and Their Quantization

Graduate lecture

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Contents

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1 Constraints and flows in the phase space

1.1 Flows generated by constraints and the geometric meaning of the Poisson bracket

□ **The set up**:

We consider a dynamical system described in a $2n$-dimensional phase space M:

$M = 2n$**-dimensional phase space**

$x^\mu = x^\mu(q^i, p_i)\,, i = 1 \sim n$: **a set of local coordinates of** M

Suppose the system is subject to a constraint defined by the equation

$$\boxed{f(x) = c = \text{constant} \qquad (1)}$$

This defines a **$(2n-1)$-dimensional hypersurface in M**, to be denoted by

Σ_f.

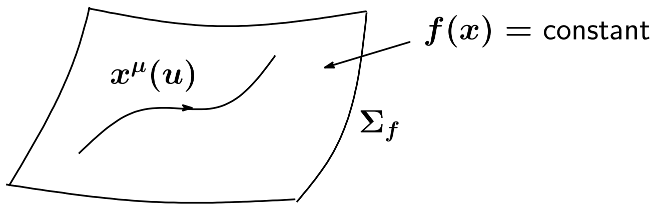
☐ A flow on Σ_f :

Consider an infinitesimal move on the surface Σ_f. Then, we have

$$f(x + dx) - f(x) = \partial_\mu f dx^\mu = 0 \tag{2}$$

Since dx^μ is tangential to Σ_f, $\partial_\mu f$ is a vector normal to Σ_f.

As we continue this development along the surface, we should get a **flow on Σ_f** of the form $x^\mu(u)$ with u a parameter of the flow.



To generate such a flow, we must make sure that dx^μ is always normal to $\partial_\mu f$.

This can be guranteed if there exists **an antisymmetric tensor $\omega^{\mu\nu}(x)$ on M**. Then we can set

$$\boxed{dx^\mu = \omega^{\mu\nu}\partial_\nu f \, du \qquad (3)}$$

This obviously satisfies the requirement $dx^\mu \partial_\mu f = 0$.

The vector field which generates this flow is defined as

$$\boxed{X_f^\omega \equiv (\omega^{\mu\nu}\partial_\nu f)\,\partial_\mu \qquad (4)}$$

so that (3) can be written as

$$dx^\mu = (X_f^\omega x^\mu) du \tag{5}$$

For a fixed $\omega^{\mu\nu}$, we will often omit the superscript ω and write X_f for short. We will call this **flow** $\mathcal{F}_f$.

□ Variation along the flow and the Poisson bracket:

Now consider how an arbitrary function $g(x)$ varies along the flow $\mathcal{F}_f$. It is given by

$$\begin{aligned} g(x+dx) &= g(x) + du X_f g(x) \\ &= g(x) + du\,\omega^{\mu\nu}\partial_\nu f \partial_\mu g \\ &= g(x) + du\,\partial_\mu g\,\omega^{\mu\nu}\partial_\nu f \\ &= g(x) + du\{g, f\}\,, \end{aligned} \tag{6}$$

where we have defined the **Poisson bracket**[1]

$$\boxed{\{g, f\} \equiv X_f g = \partial_\mu g\,\omega^{\mu\nu}\partial_\nu f = -\{f, g\} = -X_g f\,.} \tag{7}$$

[1]This should actually be called "pre-Poisson bracket". For true Poisson bracket, we need the requirements for $\omega^{\mu\nu}$ to be discussed below.

(Strictly speaking, for this to be called a Poisson bracket, $\omega^{\mu\nu}$ must satisfy certain properties to be discussed below.)

In any case, **the Poisson bracket $\{g, f\}$ describes the infinitesimal change of g along a flow $\mathcal{F}_f$ on Σ_f**. The basic formula is

$$\boxed{\frac{dg}{du} = \{g, f\} \qquad (8)}$$

□ Requirements on $\omega^{\mu\nu}$:

(1): As we have seen, once $\omega^{\mu\nu}$ is given, to a function $f(x)$ we can associate a unique flow $\mathcal{F}_f$ on the surface Σ_f.

We wish to achieve **the converse**, namely **we want a flow to determine the family of surfaces $f(x) = c$ on which it exists**. That is, we want to be able to solve

$$\frac{dx^\mu}{du} = \omega^{\mu\nu}\partial_\nu f \tag{9}$$

for $f(x)$ *up to a constant.* Obviously the condition is that **$\omega^{\mu\nu}$ is non-degenerate** (invertible).

When $\omega^{\mu\nu}$ is non-degenerate, we will denote its inverse by $\omega_{\mu\nu}$:

$$\omega^{\mu\nu}\omega_{\nu\rho} = \delta^\mu_\rho \,. \tag{10}$$

Then it is natural to define the following 2-form

$$\boxed{\begin{aligned} \omega &\equiv \tfrac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu \,, &(11)\\ \omega_{\mu\nu} &= \omega(\partial_\mu, \partial_\nu) &(12)\end{aligned}}$$

The Poisson bracket can then be written as

$$\{f, g\} = \omega(X_g, X_f) \tag{13}$$

since

$$\begin{aligned} \omega(X_g, X_f) &= \omega(\omega^{\mu\nu}\partial_\nu g \partial_\mu, \omega^{\alpha\beta}\partial_\beta f \partial_\alpha) \\ &= \omega^{\mu\nu}\partial_\nu g \omega_{\mu\alpha}\omega^{\alpha\beta}\partial_\beta f \\ &= \partial_\mu f \omega^{\mu\nu}\partial_\nu g \end{aligned} \tag{14}$$

(2): To be relevant to physical problems, $\omega_{\mu\nu}$ should not be arbitrary. **We require** that there exists a coordinate system (q^i, p_i) such that ω takes **the standard form**

$$\boxed{\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \, . \tag{15}}$$

This amounts to

$$\omega_{p_i q_j} = \delta_{ij}\,, \quad \omega_{q_j p_i} = -\delta_{ij} \tag{16}$$
$$\omega^{p_i q_j} = -\delta^{ij}\,, \quad \omega^{q_j p_i} = \delta^{ij} \tag{17}$$

Thus, in this coordinate system, the Poisson bracket takes the familiar form:

$$\boxed{\begin{aligned} \{f, g\} &= \partial_\mu f \omega^{\mu\nu} \partial_\nu g \\ &= \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) \end{aligned} \tag{18}}$$

In particular,

$$\{q^i, p_j\} = \delta^i_j \tag{19}$$

It is obvious that in this coordinate system ω is closed *i.e.* $d\omega = 0$.

• Since this is a coordinate independent concept, **we must demand that it be true in any coordinate system.**

As we shall see shortly, this will be quite essential for the Poisson bracket to have the desired properties. In tensor notation, the closedness reads

$$\boldsymbol{d\omega} = \tfrac{1}{2}\boldsymbol{\partial_\rho \omega_{\mu\nu} dx^\rho} \wedge \boldsymbol{dx^\mu} \wedge \boldsymbol{dx^\nu}\,, \tag{20}$$

$$\therefore \qquad \mathbf{0} = \boldsymbol{\partial_\rho \omega_{\mu\nu}} + \text{cyclic permutations}\,. \tag{21}$$

A non-degenerate and closed 2-form ω is called a **symplectic form** or a **symplectic structure**. An even-dimensional manifold endowed with a symplectic form is called a **symplectic manifold**.

□ Properties of the Poisson Bracket:

We now prove the following basic properties of the Poisson brackets:

$$(1) \qquad \{f, g\} = \{f, x^\mu\}\, \partial_\mu g \tag{22}$$

$$(2) \qquad \{f, g\} = -\{g, f\} \quad \Leftrightarrow \quad X_g f = -X_f g \tag{23}$$

$$(3) \qquad \{f, gh\} = \{f, g\}\, h + g\, \{f, h\} \tag{24}$$

$$(4) \qquad X_{\{g,f\}} = [X_f, X_g] \tag{25}$$

$$(5) \qquad \{f, \{g, h\}\} + \text{cyclic} = 0 \tag{26}$$

(1) and (2) are obvious from the definition. (3) is easy to prove using the vector field notation. Once (4) is proved, then (5) is automatic. So, the only non-trivial property to be proved is (4).

Proof of (4): We will give a basis independent derivation below and leave the proof in terms of components as an exercise.

(1) First it is easy to prove that for any vector field $\boldsymbol{Y}$, we have the identity

$$(*) \qquad \boldsymbol{\omega}(\boldsymbol{Y}, \boldsymbol{X}_f) = \boldsymbol{Y}(\boldsymbol{f}) \tag{27}$$

Indeed, writing $\boldsymbol{Y} = \boldsymbol{Y}^\alpha \boldsymbol{\partial}_\alpha$,

$$\begin{aligned} \boldsymbol{\omega}(\boldsymbol{Y}, \boldsymbol{X}_f) &= \boldsymbol{\omega}(\boldsymbol{Y}^\alpha \boldsymbol{\partial}_\alpha, \boldsymbol{\omega}^{\mu\nu} \boldsymbol{\partial}_\nu \boldsymbol{f} \boldsymbol{\partial}_\mu) \\ &= \boldsymbol{Y}^\alpha \boldsymbol{\omega}_{\alpha\mu} \boldsymbol{\omega}^{\mu\nu} \boldsymbol{\partial}_\nu \boldsymbol{f} = \boldsymbol{Y}^\alpha \boldsymbol{\partial}_\alpha \boldsymbol{f} = \boldsymbol{Y}(\boldsymbol{f}) \end{aligned} \tag{28}$$

(2) Next we recall the **basis-independent definition of the exterior derivative operator d**. On a $\boldsymbol{p}$-form $\boldsymbol{\omega}$,

$$\boxed{\begin{aligned} & d\boldsymbol{\omega}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_{p+1}) \\ & \quad \equiv \sum_{i=1}^{p+1} (-1)^{1+i} \boldsymbol{X}_i \left(\boldsymbol{\omega}(\boldsymbol{X}_1, \ldots, \check{\boldsymbol{X}}_i, \ldots, \boldsymbol{X}_{p+1}) \right) \\ & \qquad + \sum_{i<j} (-1)^{i+j} \boldsymbol{\omega}([\boldsymbol{X}_i, \boldsymbol{X}_j], \boldsymbol{X}_1, \ldots, \check{\boldsymbol{X}}_i, \ldots, \check{\boldsymbol{X}}_j, \ldots, \boldsymbol{X}_{p+1}) \end{aligned} \quad (29)}$$

where $\check{X}_i$ means that it be deleted.

For $p = 0$, *i.e.* when ω is just a function, one defines

$$d\omega(X) = X(\omega) \tag{30}$$

For $p = 1$ the formula takes the form

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \tag{31}$$

Let us check this by comparing it to the usual component calculation.
Set $\omega = \omega_\nu dx^\nu, X = X^\alpha \partial_\alpha, Y = Y^\beta \partial_\beta$. Then the component calculation goes as follows:

$$d\omega = \partial_\mu \omega_\nu dx^\mu dx^\nu = \tfrac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu dx^\nu \tag{32}$$

$$\begin{aligned} \therefore \qquad d\omega(X, Y) &= \tfrac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu dx^\nu (X^\alpha \partial_\alpha, Y^\beta \partial_\beta) \\ &= \tfrac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)(X^\mu Y^\nu - X^\nu Y^\mu) \end{aligned} \tag{33}$$

On the other hand, we have

$$\boldsymbol{X}(\boldsymbol{\omega}(\boldsymbol{Y})) = \boldsymbol{X}^{\alpha}\boldsymbol{\partial}_{\alpha}(\boldsymbol{\omega}_{\nu}\boldsymbol{Y}^{\nu}) = \boldsymbol{X}^{\alpha}\boldsymbol{\partial}_{\alpha}\boldsymbol{\omega}_{\nu}\boldsymbol{Y}^{\nu} + \boldsymbol{X}^{\alpha}\boldsymbol{\omega}_{\nu}\boldsymbol{\partial}_{\alpha}\boldsymbol{Y}^{\nu} \tag{34}$$

$$-\boldsymbol{Y}(\boldsymbol{\omega}(\boldsymbol{X})) = -\boldsymbol{Y}^{\beta}\boldsymbol{\partial}_{\beta}\boldsymbol{\omega}_{\nu}\boldsymbol{X}^{\nu} - \boldsymbol{Y}^{\beta}\boldsymbol{\omega}_{\nu}\boldsymbol{\partial}_{\beta}\boldsymbol{X}^{\nu} \tag{35}$$

$$\begin{aligned} -\boldsymbol{\omega}([\boldsymbol{X},\boldsymbol{Y}]) &= -\boldsymbol{\omega}((\boldsymbol{X}^{\alpha}\boldsymbol{\partial}_{\alpha}\boldsymbol{Y}^{\beta})\boldsymbol{\partial}_{\beta} - (\boldsymbol{Y}^{\beta}\boldsymbol{\partial}_{\beta}\boldsymbol{X}^{\alpha})\boldsymbol{\partial}_{\alpha}) \\ &= \boldsymbol{X}^{\alpha}\boldsymbol{\partial}_{\alpha}\boldsymbol{Y}^{\beta}\boldsymbol{\omega}_{\beta} + \boldsymbol{Y}^{\beta}\boldsymbol{\partial}_{\beta}\boldsymbol{X}^{\alpha}\boldsymbol{\omega}_{\alpha} \end{aligned} \tag{36}$$

So the last term of the RHS of (31) removes the derivatives on $\boldsymbol{X}^{\alpha}$ and $\boldsymbol{Y}^{\beta}$ and we get

$$\begin{aligned} \text{RHS of (31)} &= \boldsymbol{\partial}_{\mu}\boldsymbol{\omega}_{\nu}(\boldsymbol{X}^{\mu}\boldsymbol{Y}^{\nu} - \boldsymbol{Y}^{\mu}\boldsymbol{X}^{\nu}) \\ &= \text{LHS of (31)} \end{aligned}$$

(3) Now apply this to our 2-form $\boldsymbol{\omega}$ in the following way and impose the closed-

ness condition:

$$
\begin{aligned}
\mathbf{0} &= d\omega(Y, X_f, X_g) \\
&= Y(\omega(X_f, X_g)) - X_f(\omega(Y, X_g)) + X_g(\omega(Y, X_f)) \\
&\quad -\omega([Y, X_f], X_g) - \omega([X_f, X_g], Y) + \omega([Y, X_g], X_f)
\end{aligned}
\tag{37}
$$

where Y is an arbitrary vector field. The first term can be written in the following two ways

$$
\begin{aligned}
(i) \quad Y(\omega(X_f, X_g)) &= Y(\{g, f\}) = \omega(Y, X_{\{g,f\}}) = -\omega(X_{\{g,f\}}, Y) \\
(ii) \quad Y(\omega(X_f, X_g)) &= Y(X_f(g)) = -Y(X_g(f)) \\
&= \tfrac{1}{2}\left(Y(X_f(g)) - Y(X_g(f))\right)
\end{aligned}
$$

Forming $2 \times (ii) - (i)$ we have

$$
Y(\omega(X_f, X_g)) = Y(X_f(g)) - Y(X_g(f)) + \omega(X_{\{g,f\}}, Y)
\tag{38}
$$

The rest of the terms can be rewritten in the following way

$$\boldsymbol{X}_{\boldsymbol{f}}(\boldsymbol{\omega}(\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{g}})) = \boldsymbol{X}_{\boldsymbol{f}}(\boldsymbol{Y}(\boldsymbol{g})) \tag{39}$$

$$\boldsymbol{X}_{\boldsymbol{g}}(\boldsymbol{\omega}(\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{f}})) = \boldsymbol{X}_{\boldsymbol{g}}(\boldsymbol{Y}(\boldsymbol{f})) \tag{40}$$

$$\boldsymbol{\omega}(\left[\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{f}}\right],\boldsymbol{X}_{\boldsymbol{g}}) = \left[\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{f}}\right](\boldsymbol{g}) \tag{41}$$

$$\boldsymbol{\omega}(\left[\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{g}}\right],\boldsymbol{X}_{\boldsymbol{f}}) = \left[\boldsymbol{Y},\boldsymbol{X}_{\boldsymbol{g}}\right](\boldsymbol{f}) \tag{42}$$

Combining all the terms, many of the terms cancel and we are left with

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\omega}(\boldsymbol{X}_{\{\boldsymbol{g},\boldsymbol{f}\}},\boldsymbol{Y}) - \boldsymbol{\omega}(\left[\boldsymbol{X}_{\boldsymbol{f}},\boldsymbol{X}_{\boldsymbol{g}}\right],\boldsymbol{Y}) \\ &= \boldsymbol{\omega}(\boldsymbol{X}_{\{\boldsymbol{g},\boldsymbol{f}\}} - \left[\boldsymbol{X}_{\boldsymbol{f}},\boldsymbol{X}_{\boldsymbol{g}}\right],\boldsymbol{Y}) \end{aligned} \tag{43}$$

Since $\boldsymbol{Y}$ is arbitrary and $\boldsymbol{\omega}$ is non-degenerate, we must have $\boldsymbol{X}_{\{\boldsymbol{g},\boldsymbol{f}\}} - \left[\boldsymbol{X}_{\boldsymbol{f}},\boldsymbol{X}_{\boldsymbol{g}}\right] = \mathbf{0}$, which proves the assertion. //

Exercise: Give a proof in terms of components.

1.2 Hamiltonian vector field, gauge symmetry and gauge-fixing

□ **Hamiltonian vector field and the equation of motion**:

Consider now the Hamiltonian function $H(x)$ of a dynamical system. For a conservative system, $H(x)$ stays constant, and we can apply the above general consideration with $f(x) = H(x)$. Thus, a flow is generated by the associated **Hamiltonian vector field X_H** on the surface Σ_H. When talking about a flow generated by X_H, we normally use t as the parameter. Using the general formula (8), we obtain the **equation of motion** for any physical quantity $g(x)$

$$\boxed{\frac{dg}{dt} = X_H g = \{g, H\} \qquad (44)}$$

□ Compatibility of H and the constraint:

Now consider the situation where we have a constraint $\phi(x) = 0$, which is different from the Hamiltonian constraint.
Actually, we must consider the set of functions of the form

$$\Phi \equiv \{\chi(x)|\chi(x) = \alpha(x)\phi(x)\} \tag{45}$$

where $\alpha(x)$ is an arbitrary function non-vanishing and non-singular on the constraint surface Σ_ϕ. Then, any member of Φ vanishes on the surface Σ_ϕ and generates essentially the **same flow as** ϕ.

Now when the system develops according to the Hamiotonian $H(x)$, it should not leave Σ_ϕ. Otherwise, the imposition of the constraint would be incompatible with the Hamiltonian.
So for compatibility, we must require that after an infinitesimal time the change

of the constraint function should vanish on Σ_ϕ. This is expressed by

$$\frac{d\phi}{dt} = \{\phi, H\} \in \Phi \tag{46}$$

When this holds, we say that **$\{\phi, H\}$ is weakly zero** and denote it by $\{\phi, H\} \sim 0$.

Since the surface Σ_ϕ is generated by any member of the set Φ, we must also have

$$\forall\chi \in \Phi \qquad \{\chi, H\} \sim 0\,. \tag{47}$$

Indeed this is guaranteed: Since χ can be written as $\chi = \alpha(x)\phi(x)$, we get

$$\{\alpha\phi, H\} \;=\; \{\alpha, H\}\,\phi + \alpha\,\{\phi, H\} \sim 0\,. \tag{48}$$

Similarly, it is easy to show that

$$\boxed{\chi_1 \sim 0\,, \quad \chi_2 \sim 0 \longrightarrow \{\chi_1, \chi_2\} \sim 0 \quad (49)}$$

(Proof: $\chi_i \sim 0 \leftrightarrow \chi_i = \alpha_i \phi$. Then, $\{\alpha_1 \chi_1, \alpha_2 \chi_2\}$ clearly vanishes on Σ_ϕ.)

□ **Equivalence relation on Σ_ϕ and gauge symmetry**:

When we impose a constraint $\phi(x) = 0$, the dimension of the phase space drops by 1.
However, from the point of view of (p, q) conjugate pair, the physical degrees of freedom should drop by 2, not just by 1.

Indeed this can happen due to the following mechanism.

Let $\psi(x) \in \Phi$. Then ψ generates a flow on Σ_ϕ given by

$$dx^\mu = \{x^\mu, \psi\}\, du\,. \tag{50}$$

The important point is that flows generated by any memeber of Φ are actually the same. Let $\chi = \alpha\psi$ be another member. Then the flow generated by χ, with a parameter v, is described by the equation

$$dx^\mu = \{x^\mu, \chi\}\, dv = (\{x^\mu, \alpha\}\, \psi + \alpha\, \{x^\mu, \psi\})dv = \{x^\mu, \psi\}\, \alpha dv \tag{51}$$

This shows that, **apart from the redefinition of the parameter u, the trajectory is the same.**

Furthermore, since $\{\psi, H\} \in \Phi$, Φ as a set is invariant under time development. Thus, the shape of the flow does not change in time t.

Thus, all the points on a flow generated by Φ describe the same physical state from the point of view of the Hamiltonian dynamics and they should be identified.

Hence, the bonafide **physical phase space P** is the quotient space

$$P = \Sigma_\Phi / \sim\,, \qquad \dim P = 2n - 2\,. \tag{52}$$

where $\sim$ denotes the equivalence relation just described. Thus Φ generates **gauge transformations** on the dynamical variable x^μ in the manner

$$\boxed{\delta_\alpha x^\mu = \alpha(x)\,\{x^\mu, \phi\}}$$

□ Idea of Gauge Fixing:

Let us call our constraint $\phi^1(x)$.

Each point in P is described by a representative of Σ_{ϕ^1}.

To pick one out, one may try to intersect Σ_{ϕ^1} by another hypersurface generated by an additional constraint $\phi^2(x) = 0$.

To guarantee that this picks out a single point on the flow generated by ϕ^1, we must demand that the value of $\phi^2(x)$ changes **monotonically along the flow**

$\mathcal{F}_{\phi^1}$. This is condition is expressed by

$$\{\phi^2, \phi^1\} = \frac{d\phi^2}{du} \neq 0 \quad \text{never crosses zero} \tag{53}$$

This procedure of introducing such an additional constraint $\phi^2(x)$ is called **gauge fixing**.

1.3 Flow in the physical space and the Dirac bracket

The Poisson bracket $\{g, f\}$ describes an infinitesimal flow of g generated by the function f. (In this context, the function $f(x)$ need not be a geuine constraint of the dynamical system.)

Suppose we have genuine constraints $\phi^a = 0, (a = 1, 2)$ as above. We may still consider the flow $\{g, f\}$ generated by f.

But in general the flow gets out of the physical space $P = \Sigma_\Phi / \sim$.

We now explain a convenient way of generating a flow in the physical phase space.

It would be very nice if we can invent **a new bracket which generates a flow that stays in the physical space.** This is achieved by the so-called **the Dirac bracket**, to be deonoted by $\{g, f\}_D$.

- **First we have the following property:**

$$f(x) \text{ generates a flow on } P \Longleftrightarrow \{\phi^a, f\} = 0\,, \quad (a = 1, 2).$$

Proof: This follows immediately from $d\phi/du = \{\phi^a, f\}$. If the flow stays on P, then cleary the conditions $\phi^a = 0$ does not change and we have $\{\phi^a, f\} = 0$. Conversely, $\{\phi^a, f\} = 0$ means that along the flow $d\phi^a/du = 0$ and hence $\phi^a = 0$ conditions are preserved.

♦ **Next, the desired bracket should have the following properties:**

1. For $\boldsymbol{f(x)}$ satisfying $\{\boldsymbol{\phi^a, f}\} = \mathbf{0}$, we want

$$\{\boldsymbol{g, f}\}_{\boldsymbol{D}} = \{\boldsymbol{g, f}\} \ . \tag{54}$$

2. When $\{\boldsymbol{\phi^a, f}\} \neq \mathbf{0}$, the bracket should automatically drop the part of the movement which gets out of the constraint surface. Specifically, we want

$$\{\boldsymbol{\phi^a, f}\}_{\boldsymbol{D}} = \mathbf{0} \, . \tag{55}$$

3. The bracket should satisfy all the basic properties of the Poisson bracket.

The bracket satisfying all these requirements is given by

$$\{g, f\}_D \equiv \{g, f\} - \{g, \phi^a\} C_{ab} \{\phi^b, f\} \ , \tag{56}$$

$$C^{ab} \equiv \{\phi^a, \phi^b\} \qquad \text{anti-symmeric, non-degenerate}$$

$$C_{ab} = \text{inverse of } C^{ab} \tag{57}$$

The second term subtracts precisely the portion which gets out of the physical space. We will see this more explicitly in the next section.

In the meantime, let us check that the properties 1,2,3 are met.

- Property 1 is obvious.

- Property 2:

$$\{\phi^a, f\}_D \equiv \{\phi^a, f\} - \underbrace{\{\phi^a, \phi^b\}}_{C^{ab}} C_{bc} \{\phi^c, f\}$$
$$= \{\phi^a, f\} - \{\phi^a, f\} = 0 \qquad (58)$$

Excercise: Check the property 3

♦ Since $\{\phi^a, f\}_D = 0$ for any function $f(x)$, **we can set ϕ^a strongly (identically) to zero when we use the Dirac bracket.**

□ Meaning of the Dirac bracket and the physical Hamiltonian H_P :

We now wish to make the meaning of the Dirac bracket clearer and construct the physical Hamiltonian which directly acts on the physical phase space P. This is achieved by employing a convenient coordinate system, to be described below.

First we recall that the Poisson bracket is basis independent as seen from the expression $\{f, g\} = \omega(X_g, X_f)$. So **we may take a special coordinate system $y^\mu(x)$ where two of the new coordinates coincide with the constraints**:

$$x^\mu \longrightarrow y^\mu(x)\,, \tag{59}$$

$$\mu = \begin{cases} a & a = 1, 2 \\ i & i = 3, 4, \ldots, 2n \end{cases} \tag{60}$$

$$y^a(x) = \phi^a(x) \tag{61}$$

Further, we choose y^i such that $\omega_y^{ai} = \{y^a, y^i\} = \{\phi^a, y^i\} = 0$.
This should be possible since this condition simply says that $y^i(x)$ generates a flow on the physical space P.
Indeed if $\{y^a, y^i\} = \omega_y^{ia} \neq 0$, then redefine y^i as

$$\tilde{y}^i \equiv y^i - \omega_y^{ib} C_{bc} y^c \tag{62}$$

Then, (omitting the subscript y on ω^{ia})

$$\{\tilde{y}^i, y^a\} = \omega^{ia} - \omega^{ib} C_{bc} C^{ca} = 0 \tag{63}$$

In this coordinate system $\omega^{\mu\nu}$ takes the form

$$\omega^{\mu\nu} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & \omega^{ij} \end{pmatrix} \tag{64}$$

Now we call

U = unphysical space spanned by $\{y^a\}$
P = physical spaces spanned by $\{y^i\}$

Because of the block diagonal form of $\omega^{\mu\nu}$, we have

$$\{g, f\} = \{g, f\}_U + \{g, f\}_P \, . \tag{65}$$

It is easy to prove the identity

$$\begin{aligned} \{g, f\} &= \{g, y^\mu\} \, \omega_{\mu\nu} \, \{y^\nu, f\} && (66) \\ &= \{g, \phi^a\} \, C_{ab} \left\{\phi^b, f\right\} + \left\{g, y^i\right\} \omega_{ij} \left\{y^j, f\right\} && (67) \end{aligned}$$

Thus we can identify

$$\begin{aligned} \{g, f\}_P &= \left\{g, y^i\right\} \omega_{ij} \left\{y^j, f\right\} && (68) \\ \{g, f\}_U &= \{g, \phi^a\} \, C_{ab} \left\{\phi^b, f\right\} && (69) \end{aligned}$$

Setting $g = \phi^a$ in this identity, we find

$$\begin{aligned} \{\phi^a, f\}_P &= \left\{\phi^a, y^i\right\} \omega_{ij} \left\{y^j, f\right\} \\ &= \omega^{ai} \omega_{ij} \left\{y^j, f\right\} = 0 && (70) \\ \{\phi^a, f\}_U &= \left\{\phi^a, \phi^b\right\} \omega_{bc} \, \{\phi^c, f\} \\ &= \left\{\phi^a, \phi^b\right\} C_{bc} \, \{\phi^c, f\} \neq 0 && (71) \end{aligned}$$

So $\{\phi^a, f\}_P$ has precisely the property of the Dirac bracket. In other words, the Dirac bracket is nothing but the bracket in the physical space. Namely

$$\boxed{\begin{aligned} \{g, f\}_D &= \{g, f\} - \{g, f\}_U \\ &= \{g, f\} - \{g, \phi^a\} C_{ab} \{\phi^b, f\} \end{aligned} \qquad (72)}$$

which is exactly the definition introduced before.

♦ If we adopt this special coordinate system, it is clear that the physical Hamiltonian is

$$H_P = H(y^1 = y^2 = 0, y^i) . \tag{73}$$

and the symplectic structure to be used is

$$\omega_P \equiv \frac{1}{2}\omega_{ij} dy^i dy^j \tag{74}$$

We then get

$$\frac{dy^i}{dt} = \{y^i, H_P\}^{\omega_P} \tag{75}$$

But the same result can be obtained in any coordinate system if we use the Dirac bracket and set $\phi^a = 0$ strongly. Indeed

$$\frac{dy^a}{dt} = \{y^a, H\}_D = \{\phi^a, H\}_D = 0 \tag{76}$$

$$\begin{aligned}
\frac{dy^i}{dt} &= \left\{y^i, H\right\}_D |_{y^a=0} = \left\{y^i, y^j\right\} \omega_{ij} \left\{y^j, H\right\} |_{y^a=0} \\
&= \left\{y^i, H_P\right\}^{\omega_P}
\end{aligned} \tag{77}$$

Thus we always get correct equations of motion by using the Dirac bracket.

2 Systems with multiple constraints

2.1 Multiple constraints and their algebra

From now on, we fix the symplectic structure and take the basis $x^\mu = (q^i, p_i)$, $i = 1, 2, \ldots, n$ such that ω is of the standard form $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$.

♦ Suppose we have m independent **bosonic** constraints [2]

$$\boxed{T_\alpha(x) = 0, \qquad \alpha = 1, 2, \ldots, m \qquad (78)}$$

The constrained surface will then be $2n - m$ dimensional.

- In order for any flow generated by these constraints to remain on this surface, all the Poisson bracket among them must be weakly zero.

- Also, for these constraints to be consistent with the time evolution, theirPoisson

[2] The case where fermionic constraints are present can also be handled, but for simplicity, we shall not do that here. In what follows, we use the BFV convention and use subscripts for the constraint indices.

bracket with the Hamiltonian, which we write H_0, must vanish weakly as well.

In other words, T_α's and H_0 must satisfy an **involutive algebra** of the form

$$\boxed{\begin{aligned} \{T_\alpha, T_\beta\} &= T_\gamma U^\gamma_{\alpha\beta}\,, \qquad (79) \\ \{H_0, T_\alpha\} &= T_\beta V^\beta_\alpha\,, \qquad (80) \end{aligned}}$$

• $U^\gamma_{\alpha\beta}$ and V^β_α are in general functions of (q^i, p_i)
• Hence the above **need <u>not</u> be a Lie algebra**. (In fact for gravity this situation occurs.)

We shall call this algebra the **algebra of constraints**. In Dirac's terminology T_α's are called the **first class constraints**.

2.2 Analysis using the Action Integral

□ **Invariance of the action under the transformations generated by the constraints**:

The action for the constrained system defined above can be written in two ways:

$$(i) \qquad S\,[q,p] = \int dt\,\left(p_i \dot{q}^i - H_0\right)\Big|_{T=0} \tag{81}$$

$$(ii) \qquad S\,[q,p,\lambda] = \int dt\,\left(p_i \dot{q}^i - H_0 + \lambda^\alpha T_\alpha\right) \tag{82}$$

where in the second expression λ^α are the Lagrange multipliers.

(1) First let us check that the action (i) is invariant under **the gauge transformations generated by the constraints**.

Denoting by $\epsilon^\alpha(p,q)$ the infinitesimal local parameters, the transformations are expressed as

$$\delta q^i = \{q^i, T_\alpha\}\,\epsilon^\alpha\,, \tag{83}$$

$$\delta p_i = \{p_i, T_\alpha\}\,\epsilon^\alpha\,. \tag{84}$$

If the point (p_i, q^i) is on the constrained surface, it stays on the surface under these variations because they are generated by the constraints. The action changes by

$$\delta S = \int dt\,\left(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \delta H_0\right)\Big|_{T=0}\,. \tag{85}$$

Now for the last term,

$$\delta H_0 = \{H_0, T_\alpha\}\,\epsilon^\alpha \sim 0 \quad (\text{ on the } T = 0 \text{ surface}) \tag{86}$$

due to the constraint algebra. As for the first and the second terms,

$$\delta p_i \dot{q}^i = \{p_i, T_\alpha\} \epsilon^\alpha \dot{q}^i = -\frac{\partial T_\alpha}{\partial q^i} \dot{q}^i \epsilon^\alpha \tag{87}$$

$$p_i \delta \dot{q}^i = -\dot{p}_i \left\{q^i, T_\alpha\right\} \epsilon^\alpha = -\frac{\partial T_\alpha}{\partial p_i} \dot{p}_i \epsilon^\alpha , \tag{88}$$

$$\therefore \qquad \delta p_i \dot{q}^i + p_i \delta \dot{q}^i = -\dot{T}_\alpha \epsilon^\alpha = -\frac{d}{dt}(T_\alpha \epsilon^\alpha) + \underbrace{T_\alpha}_{0} \dot{\epsilon}^\alpha \tag{89}$$

(In the second equation above, we used the integration by parts under $\int dt$.)
So for the variations that vanish at the initial and the final time this vanishes upon integration on the $T = 0$ surface.

Thus S is invariant. Since the classical trajectories are obtained by extremizing the action, this shows that **various trajectories which differ by the variations generated by the constraints all satisfy the equations of motion**.

(2) Next consider the variation of $S[q, p, \lambda]$. **This time, we cannot use the equations $T_\alpha = 0$.**

Using the previous results and the integration by parts, we get

$$\begin{aligned}
\delta p_i \dot{q}^i + p_i \delta \dot{q}^i &= -\dot{T}_\alpha \epsilon^\alpha = T_\alpha \dot{\epsilon}^\alpha \,, & (90)\\
-\delta H_0 &= -\left\{ H_0, T_\alpha \right\} \epsilon^\alpha = -T_\alpha V^\alpha_\beta \epsilon^\beta \,, & (91)\\
\delta(\lambda^\alpha T_\alpha) &= \delta\lambda^\alpha T_\alpha + \lambda^\alpha \left\{ T_\alpha, T_\beta \right\} \epsilon^\beta \\
&= \delta\lambda^\alpha T_\alpha + \lambda^\alpha T_\gamma U^\gamma_{\alpha\beta} \epsilon^\beta \,. & (92)
\end{aligned}$$

So $\delta S\,[q, p, \lambda]$ vanishes if we define the variation of λ^α as

$$\delta\lambda^\alpha = -\dot{\epsilon}^\alpha + V^\alpha_\nu \epsilon^\nu - \lambda^\beta U^\alpha_{\beta\gamma} \epsilon^\gamma \,. \qquad (93)$$

It should be noted that **in the case of the usual gauge theory this is precisely of the form of the familiar gauge transformation** .

(for $\lambda^\alpha \sim A^a_0$ with $V^\alpha_\nu = 0$) .

☐ Gauge Fixing:

Just like in the case of a single constraint, we add m additional constraints to pick out the true physical phase space:

$$\Theta^\alpha(p_i, q^i) \;=\; 0 \qquad \alpha = 1, 2, \ldots, m\,. \tag{94}$$

Since the new surface must intersect the original constraint sufrace we need to require

$$\delta\Theta^\alpha \;=\; \{\Theta^\alpha, T_\beta\}\,\epsilon^\beta \neq 0 \qquad {}^\forall\alpha, {}^\forall\epsilon^\beta\,. \tag{95}$$

so that on any tranjectory the value of Θ^α must be changing.

In other words, the matrix $\{\Theta^\alpha, T_\beta\}$ need not have zero eigenvalue. Thus we demand

$$\det\,\{\Theta^\alpha, T_\beta\} \neq 0 \tag{96}$$

Note that Θ^α's do not form any involutive algebra.

Let us collect all the $2m$ constraints and define

$$\boxed{\begin{aligned} \mathcal{T}_a &= \begin{pmatrix} T_\alpha \\ \Theta^\beta \end{pmatrix} , & (97) \\ C_{ab} &= \{\mathcal{T}_a, \mathcal{T}_b\} . & (98) \end{aligned}}$$

Then by looking at the structure of C_{ab} one can show that $\det C \neq 0$.

Excercise Prove this fact.

To enforce the additional constraints, we introduce m Lagrange multipliers $\bar{\lambda}_\beta$ for Θ^β. The action can be written as

$$S\left[q, p, \lambda, \bar{\lambda}\right] = \int dt \left(p_i \dot{q}^i - H_0 + \lambda^\alpha T_\alpha + \bar{\lambda}_\alpha \Theta^\alpha\right) \tag{99}$$

$$= \int dt \left(p_i \dot{q}^i - H_0 + \xi^a \mathcal{T}_a\right) , \tag{100}$$

where $$\xi^a = \begin{pmatrix} \lambda^\alpha \\ \bar{\lambda}_\beta \end{pmatrix} \tag{101}$$

The dimension of the physical space is $2(n-m)$.

□ Equations of motion and emergence of the Dirac bracket:

Variations with respect to q, p and ξ lead to the following equations of motion:

$$\delta p_i \;:\qquad 0 = \dot{q}^i - \frac{\partial H_0}{\partial p_i} + \frac{\partial \mathcal{T}_a}{\partial p_i}\xi^a\,, \tag{102}$$

$$\delta q^i \;:\qquad 0 = -\dot{p}^i - \frac{\partial H_0}{\partial q^i} + \frac{\partial \mathcal{T}_a}{\partial q^i}\xi^a\,, \tag{103}$$

$$\delta \xi^a \;:\qquad 0 = \mathcal{T}_a(p,q)\,. \tag{104}$$

Since the last condition should not change in time, we have

$$0 \;=\; \frac{d\mathcal{T}_a}{dt} = \dot{p}_i\frac{\partial \mathcal{T}_a}{\partial p_i} + \dot{q}^i\frac{\partial \mathcal{T}_a}{\partial q^i}\,. \tag{105}$$

Substituting the expressions for $\dot{p}_i$ and $\dot{q}^i$, this becomes

$$\begin{aligned} 0 &= \left(-\frac{\partial H_0}{\partial q^i} + \frac{\partial \mathcal{T}_b}{\partial q^i}\xi^b\right)\frac{\partial \mathcal{T}_a}{\partial p_i} + \left(\frac{\partial H_0}{\partial p_i} - \frac{\partial \mathcal{T}_b}{\partial p_i}\xi^b\right)\frac{\partial \mathcal{T}_a}{\partial q^i} \\ &= -\{H_0, \mathcal{T}_a\} - \{\mathcal{T}_a, \mathcal{T}_b\}\,\xi^b \\ &= -\{H_0, \mathcal{T}_a\} - C_{ab}\xi^b\,. \end{aligned} \tag{106}$$

Since the matrix C_{ab} is invertible we can solve for ξ^a *uniquely* and get

$$\xi^a = C^{ab}\{\mathcal{T}_b, H_0\}\ . \tag{107}$$

Putting this back into the equations of motion, we find

$$\begin{aligned} \dot{q}^i &= \{q^i, H_0\} - \{q^i, \mathcal{T}_a\}\,C^{ab}\{\mathcal{T}_b, H_0\} \\ &= \{q^i, H_0\}_D\ , \end{aligned} \tag{108}$$

$$\begin{aligned} \dot{p}_i &= \{p_i, H_0\} - \{p_i, \mathcal{T}_a\}\,C^{ab}\{\mathcal{T}_b, H_0\} \\ &= \{p_i, H_0\}_D\ . \end{aligned} \tag{109}$$

Therefore, **the equations of motion are generated precisely by the Dirac bracket**.

3 Dirac's theory of constrained systems

Up until now we have been considering constrained Hamiltonian systems. We now study how constraints arise in Lagrangian formulation.

Questions:

- When do we get constraints?
- How to find constraints systematically?

Dirac's theory gives the answer.

3.1 Singular Lagrangian and primary constraints

□ **Singular Lagrangian**:

Consider a Lagrangian of the form

$$L = \tfrac{1}{2}\dot{q}^i a_{ij}(q)\dot{q}^j + \dot{q}^i b_i(q) - V(q) \tag{110}$$

$$a_{ij} = a_{ji} \quad \text{symmetric} \tag{111}$$

The momentum conjugate to q^i is

$$p_i = a_{ij}(q)\dot{q}^j + b_i(q)\,, \tag{112}$$

$$\text{or} \qquad a_{ij}(q)\dot{q}^j = p_i - b_i(q)\,. \tag{113}$$

When the matrix $a_{ij}(q)$ is singular, then $\dot{q}$ cannot be solved in terms of p uniquely. Such a Lagrangian is called **singular**.

In this case, $a_{ij}(q)$ has zero eigenvalues. Let v_a^i $(a = 1, 2, \ldots, m_1)$ be zero eigenvectors. Then,

$$0 = v_a^i a_{ij}(q) = v_a^i\,(p_i - b_i(q)) \equiv \phi_a(p, q)\,, \tag{114}$$

and we have so called the **primary constraints**.

♦ They arise entirely from the definition of the momenta and therefore may not be compatible with the equations of motion.

Special but important case of a singular Lagrangian:

First order system: It may sometimes happen that $a_{ij} = 0$ so that there are no quadratic kinetic term. In this case, we get the primary constraints of the form

$$p_i - b_i(q) = 0 \tag{115}$$

This occurs for the Dirac equation, which therefore is a constrained system. We will describe how it should be properly quantized later.

□ **Canonical Hamiltonian**:

Lte us construct the Hamiltonian in the usual way:

$$H_{can} = p_i \dot{q}^i - L(q, \dot{q}) \tag{116}$$

It is called the **canonical Hamiltonian**. As is well known, H_{can} is a function only of p and q and does not depend on $\dot{q}$. To check this, take the variation:

$$\begin{aligned} \delta H_{can} &= \delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \\ &= \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \,. \end{aligned} \tag{117}$$

Hence $\delta \dot{q}^i$ does not appear in δH_{can} and we have $\partial H_{can}/\partial \dot{q}^i = 0$.

3.2 Compatibility of the primary constraints and the Hamiltonian

Previously, we assumed the compatibility of the constraints ϕ_a with the Hamiltonian H , which is expressed as

$$\{\phi_a, H\} \sim 0 \,. \tag{118}$$

If it is not automatically met, we have to impose these conditions for a consistent theory.

This in general produces further constraints.

To perform this analysis, we must note that because of the existence of the primary constraints the following Hamiltonian, called the **total Hamiltonian**, is as good as the canonical one:

$$\boxed{\boldsymbol{H_T = H_{can} + \lambda^a(p,q)\phi_a(p,q)} \quad (119)}$$

In fact, its variation $\boldsymbol{\delta H_T}$ is identical to $\boldsymbol{\delta H_{can}}$ because $\boldsymbol{\phi_a = \delta\phi_a = 0}$[3] and the solutions of the equations of motions derived from $\boldsymbol{H_T}$ for any $\boldsymbol{\lambda^a}$ extremize the action. The equations of motion read, after setting $\boldsymbol{\phi_a = 0}$,

$$\boldsymbol{\dot{q}^i = \frac{\partial H_{can}}{\partial p_i} + \lambda^a \frac{\partial \phi_a}{\partial p_i}}, \qquad (120)$$

$$\boldsymbol{\dot{p}_i = -\frac{\partial H_{can}}{\partial q^i} - \lambda^a \frac{\partial \phi_a}{\partial q^i}}. \qquad (121)$$

Remembering that Poisson brackets are computed without enforcing constraints,

[3]Since $\boldsymbol{\phi_a = 0}$ should be enforced, its variation must also be zero. In other words, the variation must be such that $\boldsymbol{\delta\phi_a = 0}$ must hold.

we can write this as

$$\dot{q}^i = \{q^i, H_{can}\} + \lambda^a \{q^i, \phi_a\} \ , \tag{122}$$
$$\dot{p}_i = \{p_i, H_{can}\} + \lambda^a \{p_i, \phi_a\} \ . \tag{123}$$

In the sense of weak equality, this can be written also as

$$\boxed{\begin{aligned} \dot{q}^i &\sim \{q^i, H_T\} \ , && (124) \\ \dot{p}^i &\sim \{p_i, H_T\} \ . && (125) \end{aligned}}$$

Compatibility with H_T:

Now we check the compatibility, *i.e.* , the invariance of the constraint surface, call it Σ_1, as the system evolves according to H_T. This reads

$$(*) \qquad 0 \sim \{\phi_a, H_T\} = \{\phi_a, H_{can}\} + \lambda^b \{\phi_a, \phi_b\} \ . \tag{126}$$

Let the **rank of $\{\phi_a, \phi_b\}$ on Σ_1 be r_1.** Then by forming appropriate linear combinations of $\{\phi_a\}$, we can bring it to the form

$$\{\phi_a, \phi_b\} = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \tag{127}$$

$$\phi_a = (\phi_\alpha, \phi_A), \tag{128}$$

$$\alpha = 1, 2, \ldots, r_1, \quad A = r_1 + 1, \ldots, m_1 \tag{129}$$

(i) For the subspace corresponding to the subscript α, we can use the inverse of $C_{\alpha\beta}$ to solve for the mulitiplier from $(*)$:

$$\lambda^\alpha = -C^{\alpha\beta}\{\phi_\beta, H_{can}\} \tag{130}$$

$$(C^{\alpha\beta} = (C^{-1})_{\alpha\beta}) \tag{131}$$

(ii) For the subspace corresponding to the subscript A, $(*)$ demands the compatibility

$$\boxed{\{\phi_A, H_{can}\} \sim 0\,. \quad (132)}$$

LHS **can** contain parts which cannot be written as a linear combination of primary constraints.

Then we must impose certain number, m_2 of **secondary constraints**. Constraint surface now becomes a more restricted one Σ_{1+2}.

At this stage, **primary and the secondary constraints must be regarded on the same footing** and we **re-set**

$$\boxed{\phi_a = \text{all the constraints} \qquad a = 1, 2, \ldots, m_1 + m_2\,, \quad (133)}$$

and **repeat the analysis again** with of course new H_T.

Continue until no new constraints are generated.

- For a system with a finite degrees of freedom, this process obviously terminates after a finite number of steps.
- In the case of field theory, where we have infinite degrees of freedom, it may require infinite steps. However, we only need finite steps if the Poisson brackets always contain $\delta(\vec{x} - \vec{y})$ so that we only get local constraints.

♦ We then end up with m constraints

$$\begin{aligned}
\phi_a \quad &: \quad a+1,2,\ldots,m && (134)\\
m \;&=\; m_1+m_2+\cdots \le 2n && (135)\\
&\quad (2n = \text{dim. of the original phase space})\\
r \;&=\; r_1+r_2+\cdots \le m\,. && (136)
\end{aligned}$$

Using (130), the total Hamiltonian will be of the form

$$\boxed{H_T = H_{can} - \phi_\alpha C^{\alpha\beta}\{\phi_\beta, H_{can}\} + \phi_A\lambda^A} \tag{137}$$

It is easy to check that for all the constraints, Poisson bracket with H_T produces a linear combination of constraints and hence weakly vanish:

$$\{\phi_a, H_T\} = \phi_b V_a^b \sim 0\,. \tag{138}$$

3.3 1st and 2nd class functions and quantization procedure

It is important to introduce the notion of **the first and the second class functions (constraints)**.

A function $\boldsymbol{R(q,p)}$ is classified as either 1st class or 2nd class according to:

$$\begin{aligned}
\textbf{1st class} &\Longleftrightarrow \{\boldsymbol{R}, \boldsymbol{\phi_a}\} \sim \boldsymbol{0} \quad \text{for all constraints } \boldsymbol{\phi_a} \\
&\Longleftrightarrow \{\boldsymbol{R}, \boldsymbol{\phi_a}\} = \boldsymbol{\phi_b r_a^{\ b}} &(139) \\
\textbf{2nd class} &\Longleftrightarrow \text{otherwise} &(140)
\end{aligned}$$

Then we have

Theorem:

$$\boldsymbol{R, S:} \quad \textbf{1st class} \Longrightarrow \{\boldsymbol{R, S}\}\boldsymbol{:} \quad \textbf{1st class} \tag{141}$$

Proof is easy using the Jacobi identity for Poisson bracket. Thus the 1st class constraints form a **closed (involutive) algebra** under Poisson bracket operation.

□ **Quantization with the second class constraints**:

There are several methods:

♦ **(1)** Solve the second class constraints explicitly and eliminate unphysical coordinates.

However, in general this is difficult and spoils manifest symmetries.

♦ **(2)** Use the Dirac bracket (as discussed in Chapter 2) and replace it with the quantum bracket in the following way:

$$\textbf{quantum bracket} = [q^i, p_j] \equiv i\hbar \{q^i, p_j\}_D \tag{142}$$

♦ **(3)** Use path-integral formalism as already discussed.

□ **Quantization with the first class constraints**:

Since 1st class constraints generate gauge transformations, we must either fix the gauge or select gauge invariant physical states by imposing these constraints.

More explicitly,

1. **Add gauge-fixing constraints** to **make them second class** and then use the methods above.

2. Replace the usual Poisson bracket with quantum bracket (with appropriate $i\hbar$ factor) and then **impose them on physical states.**

$$\boxed{\phi_A|\Psi\rangle = 0 \qquad (143)}$$

In this approach, **operator ordering** is an important problem.
For compatibility, we must have

$$[\phi_a, \phi_b]\,|\Psi\rangle = 0 \tag{144}$$

and for this purpose, one must find an ordering such that

$$[\phi_a, \phi_b] = U^c_{ab}\phi_c \tag{145}$$

holds *i.e.* ϕ_c appears to the right of U^c_{ab}.

When there does not exist any such ordering, then the system becoms inconsistent and it is said to possess **quantum commutator anomaly**.

3.4 Application to abelian gauge theory

We now apply the Dirac's theory to the Maxwell field and see how it works.

☐ **Analysis of constraints**:

Lagrangian:

$$\boldsymbol{L = -\frac{1}{4}\int d^3x F_{\mu\nu}F^{\mu\nu}} \tag{146}$$

$$\boldsymbol{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{147}$$

Momentum $\boldsymbol{\Pi^\mu}$ conjugate to $\boldsymbol{A_\mu}$:

Vary with respect to $\dot{A}_\mu$:

$$\begin{aligned} \delta L &= -\frac{1}{2}\int d^3x F^{\mu\nu}\delta F_{\mu\nu} \\ &\ni -\int d^3x F^{\mu 0}\delta F_{\mu 0} \ni \int d^3x F^{\mu 0}\delta \dot{A}_\mu \end{aligned} \tag{148}$$

Therefore we get

$$\boxed{(\star) \qquad \Pi^\mu = F^{\mu 0} \qquad (149)}$$

Equal time Poisson bracket:

$$\{A_\mu(x), \Pi^\nu(y)\} = \delta^\nu_\mu \delta(x-y) \tag{150}$$

where x means $\vec{x}$ and the δ-function is the 3-dimensional one.

Now from $(\star)$ above, we get the following primary constraint since $F^{00} = 0$:

$$\Pi^0 = 0 \tag{151}$$

Form of $\boldsymbol{H_{can}}$:

$$\boldsymbol{H_{can} = \int d^3x \Pi^\mu A_{\mu,0} - L}$$

$$\boldsymbol{= \int d^3x \left(\underbrace{F^{i0} A_{i,0}}_{(*)} + \frac{1}{4} F^{ij} F_{ij} + \tfrac{1}{2} F^{i0} F_{i0} \right)} \qquad (152)$$

Rewrite $(*)$:

$$\boldsymbol{(*) = F^{i0}(A_{i,0} - A_{0,i}) + F^{i0} A_{0,i} = -F^{i0} F_{i0} + F^{i0} A_{0,i}} \qquad (153)$$

Putting this back in and using the definition of the momentum we get

$$\boldsymbol{H_{can} = \int d^3x \left(\frac{1}{4} F^{ij} F_{ij} + \tfrac{1}{2} \Pi^i \Pi^i - A_0 \partial_i \Pi^i \right)} \qquad (154)$$

Compatibility of $\boldsymbol{\Pi^0 = 0}$ with $\boldsymbol{H_{can}}$:

We must demand $\boldsymbol{\{\Pi^0, H_{can}\} \sim 0}$. This immediately gives the **Gauss law constraint** as **a secondary constraint**:

$$\boxed{\mathcal{G} \equiv \partial_i \Pi^i = 0 \qquad (155)}$$

Compatibility of $\mathcal{G} = 0$ with H_{can}:

Prepare some formulas

$$\{A_0(x), \mathcal{G}(y)\} = 0 \tag{156}$$

$$\{\Pi^0(x), \mathcal{G}(y)\} = 0 \tag{157}$$

$$\{A_i(x), \mathcal{G}(y)\} = \frac{\delta}{\delta \Pi^i(x)} \mathcal{G}(y) = \partial_i^y \delta(x - y) \tag{158}$$

$$\{\Pi^i(x), \mathcal{G}(y)\} = -\frac{\delta}{\delta A_i(x)} \mathcal{G}(y) = 0 \tag{159}$$

$$\{\mathcal{G}(x), \mathcal{G}(y)\} = 0 \tag{160}$$

♦ Note that the second equation tells us that the multiplier λ in the term $\lambda\mathcal{G}$ in H_T is not determined.

$\boldsymbol{H_T}$ is given by

$$\boldsymbol{H_T = H_{can} + \lambda \mathcal{G}} \tag{161}$$

Taking the Poisson bracket with $\boldsymbol{\Pi^0}$, we get

$$\boldsymbol{\{\Pi^0, H_T\} = \{\Pi^0, H_{can}\} + \lambda \cancel{\{\Pi^0, \mathcal{G}\}}} \tag{162}$$

so that $\boldsymbol{\lambda}$ is not determined.

♦ The 3rd equation expresses the gauge transformation of $\boldsymbol{A_i}$. Indeed, introducing the gauge parameter $\boldsymbol{\Lambda(y)}$, we have

$$\begin{aligned}\boldsymbol{\left\{A_i(x), \int \mathcal{G}(y)\Lambda(y)\right\}} &= \boldsymbol{\int dy \Lambda(y) \partial_i^y \delta(x-y)} \\ &= \boldsymbol{-\int dy \partial_i \Lambda(y) \delta(x-y) = -\partial_i \Lambda(x)}\end{aligned} \tag{163}$$

Now let us compute $\{H_{can}, \mathcal{G}(y)\}$ making use of these formulas.

$$\{H_{can}, \mathcal{G}(y)\} = \int d^3x \left\{ \frac{1}{4} F^{jk} F_{jk}(x) + \tfrac{1}{2} \Pi^i \Pi^i(x), \mathcal{G}(y) \right\}$$
$$= \int d^3x \tfrac{1}{2} F^{jk}(x) \{F_{jk}(x), \mathcal{G}(y)\} \quad (164)$$

But

$$\{F_{jk}(x), \mathcal{G}(y)\} = \{\partial_j A_k(x) - \partial_k A_j(x), \mathcal{G}(y)\}$$
$$= \partial_j^x \{A_k(x), \mathcal{G}(y)\} - \partial_k^x \{A_j(x), \mathcal{G}(y)\}$$
$$= \partial_j^x \partial_k^y \delta(x-y) - \partial_k^x \partial_j^y \delta(x-y)$$
$$= -\partial_j^x \partial_k^y \delta(x-y) + \partial_k^x \partial_j^x \delta(x-y) = 0 \quad (165)$$

Thus we have $\{H_{can}, \mathcal{G}(y)\} = 0$ identically and no new constraints are generated. So actually H_{can} is already H_T.

□ Coulomb gauge-fixing and the Dirac bracket:

Let us denote the two 1st class constraints by

$$\phi_1(x) = \Pi^0(x) = 0 \tag{166}$$

$$\phi_3(x) = \mathcal{G}(x) = 0 \tag{167}$$

Let us take the **Coulomb gauge** by adding the following two additional constraints:

$$\phi_2(x) = A_0(x) = 0 \tag{168}$$

$$\phi_4(x) = \partial_i A^i(x) = 0 \tag{169}$$

Non-vanishing Poisson brackets among them are

$$\{\phi_1(x), \phi_2(x)\} = -\delta(x-y) \tag{170}$$

$$\begin{aligned}\{\phi_3(x), \phi_4(x)\} &= \left\{\partial_i \Pi^i(x), \partial_j A^j(y)\right\} \\ &= \partial_i^x \partial_j^y \left\{\Pi^i(x), A^j(y)\right\} = \partial_i^x \partial_j^y (-\delta(x-y)) \\ &= \partial_x^2 \delta(x-y) \end{aligned} \tag{171}$$

Therefore

$$\boldsymbol{C_{ab}(x,y) = \{\phi_a(x), \phi_b(y)\} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}} \tag{172}$$

where
$$\boldsymbol{A = \begin{pmatrix} 0 & -\delta(x-y) \\ \delta(x-y) & 0 \end{pmatrix}} \tag{173}$$

$$\boldsymbol{B = \begin{pmatrix} 0 & \partial_x^2\delta(x-y) \\ -\partial_x^2\delta(x-y) & 0 \end{pmatrix}} \tag{174}$$

It's inverse is given by

$$\boldsymbol{C^{ab}(x,y) = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}}$$

$$\boldsymbol{A^{-1} = \begin{pmatrix} 0 & \delta(x-y) \\ -\delta(x-y) & 0 \end{pmatrix} = -A} \tag{175}$$

$$\boldsymbol{B^{-1} = \begin{pmatrix} 0 & -D(x-y) \\ D(x-y) & 0 \end{pmatrix}} \tag{176}$$

where
$$\boldsymbol{\partial_x^2 D(x-y) = \delta(x-y)} \tag{177}$$

Now we can compute the basic Dirac bracket:

$$\begin{aligned}
\{A^i(x), \Pi_j(y)\}_D &= \delta^i_j \delta(x-y) \\
&\quad - \int d^3u d^3v \,\{A^i(x), \phi_a(u)\}\, C^{ab}(u,v)\, \{\phi_b(v), \Pi_j(y)\} \\
&= \delta^i_j \delta(x-y) \\
&\quad - \int d^3u d^3v \,\{A^i(x), \phi_3(u)\}\, C^{34}(u,v)\, \{\phi_4(v), \Pi_j(y)\} \\
&= \delta^i_j \delta(x-y) \\
&\quad - \int d^3u d^3v \,\left(-\partial^i_x \delta(x-u)\right) \left(-D(u-v)\right) \left(-\partial^y_j \delta(v-y)\right) \\
&= \delta^i_j \delta(x-y) - \delta^i_x \delta^x_j D(x-y) \\
&= \left(\delta^i_j - \frac{\partial^i \partial_j}{\partial^2}\right) \delta(x-y) = \text{transverse } \delta\text{-function}
\end{aligned} \tag{178}$$

We can check that the gauge condition and the Gauss law are satisfied due to the transversality of the δ-function:

$$\partial_i^x \left\{A^i(x), \Pi_j(y)\right\}_D = 0 \tag{179}$$

$$\partial_y^j \left\{A^i(x), \Pi_j(y)\right\}_D = 0 \tag{180}$$

4 More general formulation of Batalin, Fradkin and Vilkovisky

4.1 General gauge fixing, including relativistic gauges

A big problem with the formulation using the Dirac bracket:

One cannot handle relativistic gauge fixing.

Example : **Lorentz gauge in QED**. The constraint is

$$\Theta = \partial_\mu A^\mu = \dot{A}^0 + \partial_i A^i = 0\,. \tag{181}$$

This involves **a time derivative of a mulitiplier A^0.**

(Recall that no time derivative of A_0 appears in the Lagrangian and, when the Hamiltonian is formed, A_0 appears as the Lagrange multiplier for the Gauss law constraint.)

Thus, in a relativistic formulation, we want to consider a **general gauge fixing function** of the form

$$\boxed{\Theta^\alpha = \Theta^\alpha(q, p, \lambda, \dot{\lambda}, \bar{\lambda}) \,, \quad (182)}$$

where λ **corresponds to** A^0 **and we may want to take** $\bar{\lambda}_\alpha$ **to be the mulitiplier for** Θ^α **itself**.

So one writes down the action

$$S\left[q, p, \lambda, \bar{\lambda}\right] = \int dt \left(p_i \dot{q}^i - H_0 + \lambda^\alpha T_\alpha(q, p) + \bar{\lambda}_\alpha \Theta^\alpha(q, p, \lambda, \dot{\lambda}, \bar{\lambda})\right) . \tag{183}$$

But in this form, in general $\bar{\lambda}_\alpha$ **is no longer a multiplier since it may appear also in** Θ^α**.**

More crucially, λ^α and $\bar{\lambda}_\alpha$ $(\alpha = 1 \sim m)$ can become conjugate to each other. In such a case, the dimension of the phase space becomes $2(n+m)$ and **not the correct value $2(n-m)$.** ($\boldsymbol{2n}$ =the original dim. of the phase space.)
We must consider **a mechanism to kill $4m$ excess degrees of freedom.**

4.2 Introduction of the ghost system

For definiteness, we shall deal with the gauge fixing constraint of the form

$$\boxed{\boldsymbol{\Theta^\alpha = \dot{\lambda}^\alpha + \mathcal{F}^\alpha(q, p, \lambda, \bar{\lambda})}\,. \quad (184)}$$

The action then becomes

$$\boldsymbol{S\left[q, p, \lambda, \bar{\lambda}\right] = \int dt \left(p_i \dot{q}^i + \bar{\lambda}_\alpha \dot{\lambda}^\alpha - H_0 + \lambda^\alpha T_\alpha + \bar{\lambda}_\alpha \mathcal{F}^\alpha\right)} \tag{185}$$

Note that clearly the $\bar{\lambda}_\alpha$ are momenta conjugate to λ_α.

Now group various functions in the following way:

$$q^A = (q^i, \lambda^\alpha)\,, \qquad A = 1, 2, \ldots, n+m \tag{186}$$
$$p_A = (p_i, \bar{\lambda}_\alpha)\,, \tag{187}$$
$$G_a = (T_\alpha, i\bar{\lambda}_\alpha)\,, \tag{188}$$
$$\chi^a = (i\lambda^\alpha, \mathcal{F}^\alpha)\,. \tag{189}$$

Then the action can be written more compactly as

$$\boxed{S\left[q^A, p_A\right] = \int dt\left(p_A \dot{q}^A - H_0 - iG_a\chi^a\right) \quad (190)}$$

To kill the unwanted degrees of freedom, **we introduce $2m$ fermionic ghost-anti-ghost conjugate pairs**:

$$(\eta^a, \wp_a)\,, \qquad a = 1, 2, \ldots, 2m\,, \tag{191}$$

$$\{\eta^a, \wp_b\} = \delta^a_b\,, \qquad \{\eta^a, \eta^b\} = \{\wp_a, \wp_b\} = 0 \tag{192}$$

Later, we will often use the further decomposition of $(\eta^a, \wp_a)$,

$$\eta^a = (\eta^\alpha, \tilde{\eta}^\alpha)\,, \qquad \alpha = 1 \sim m$$
$$\wp_a = (\wp_\alpha, \tilde{\wp}_\alpha)$$

□ Graded Poisson bracket:

To deal with the fermionic degrees of freedom, we need to generalize the concept of Poisson bracket. The general definition of graded-Poisson bracket is

$$\{F, G\} \equiv (\partial F/\partial Q^A)(\partial/\partial P_A)G - (-1)^{|F||G|}(\partial G/\partial Q^A)(\partial/\partial P_A)F\,,$$
$$|F| = \begin{cases} 0 & F \text{ is bosonic} \\ 1 & F \text{ is fermionic} \end{cases}, \tag{193}$$

where $(\partial F/\partial Q^A)$ represents the right-derivative $F\frac{\overleftarrow{\partial}}{\partial Q^A}$. So more explicit representation is

$$\boxed{\{F, G\} \equiv F\frac{\overleftarrow{\partial}}{\partial Q^A}\frac{\overrightarrow{\partial}}{\partial P_A}G - (-1)^{|F||G|}G\frac{\overleftarrow{\partial}}{\partial Q^A}\frac{\overrightarrow{\partial}}{\partial P_A}F \qquad (194)}$$

(Q^A, P_A) denotes all the conjugate pairs including ghosts, λ^α and $\bar{\lambda}_\alpha$. This definition is consistent with the Poisson brackets for η^a and $\wp_b$ given above.

An important property of the graded Poisson bracket is the **graded Jacobi identity**. It can be written as

$$\{A, \{B, C\}\} = \{\{A, B\}, C\} + (-1)^{|A||B|}\{B, \{A, C\}\} . \tag{195}$$

Exercise: Check this property. (It is rather non-trivial.)

4.3 Batalin-Vilkovisky theorem

Now we modify our action by adding $\int dt(\wp_a\dot{\eta}^a - \Delta H)$:

$$S = \int dt \left(p_A\dot{q}^A + \wp_a\dot{\eta}^a - H\right) , \tag{196}$$

$$H = H_0 + iG_a\chi^a + \Delta H . \tag{197}$$

The problem is to find the appropriate ΔH which makes this system equivalent to the canonical system described solely in terms of the physical degrees of freedom.

The **basic theorem** for solving this problem is the following due to Batalin and Vilkovisky:

4.3.1 BV Theorem

Assume that the set of functions $\boldsymbol{G_a}$ are **algebraically independent**[4] and satisfy the involutive algebra:

$$\begin{aligned} \{G_a, G_b\} &= G_c U^c_{ab} \qquad (198) \\ \{H_0, G_a\} &= G_b V^b_a \qquad (199) \end{aligned}$$

Let $\Psi(q^A, p_A, \eta^a, \wp_a)$ be **an arbitrary fermionic function**, to be called **gauge fermion** (fermionic gauge-fixing function).

Then the following functional integral is **independent of the choice of Ψ:**

[4] It means that $\sum_a G_a A_a = 0 \quad \Rightarrow A_a = 0$.

$$Z_\Psi = \int dq\,dp\,d\eta\,d\wp\,\mathrm{e}^{iS_\Psi} \tag{200}$$

$$S_\Psi = \int dt\,\left(p_A\dot{q}^A + \wp_a\dot{\eta}^a - H_\Psi\right) \tag{201}$$

$$H_\Psi = H_0 + \wp_a V^a_b\eta^b + i\{\Psi, \Omega\}\,, \tag{202}$$

$$\Omega \equiv G_a\eta^a + \tfrac{1}{2}(-1)^{|a|}\wp_a U^a_{bc}\eta^c\eta^b\,. \tag{203}$$

Ω is called the **BRST operator**.

(Be careful about the order of the indices of last two η's.)

♦ The definition above is **valid for a mixed system of bosonic and fermionic constraints G_a**.

$|a|$ is 0 if G_a is bosonic and is 1 if G_a is fermionic. Accordingly, the ghosts are fermionic for the former and bosonic for the latter.

Remark: For the case at hand with $\boldsymbol{G_a = (T_\alpha, i\bar{\lambda}_\alpha)}$, the involutive algebra is extended. This extension however is rather trivial since $\boldsymbol{T_\alpha}$ and $\boldsymbol{H_0}$ are assumed to be functions only of $\boldsymbol{(q, p)}$ and hence $\boldsymbol{i\bar{\lambda}_\alpha}$ is completely inert.

Typical example of Ψ: The often used form of $\boldsymbol{\Psi}$ is

$$\boxed{\boldsymbol{\Psi = \wp_a \chi^a = i\wp_\alpha \lambda^\alpha + \tilde{\wp}_\alpha \mathcal{F}^\alpha} \quad (204)}$$

Then by computing $\boldsymbol{\{\Psi, \Omega\}}$, we get

$$H_\Psi = H_0 + iG_a\chi^a + \Delta H = H_0 - T_\alpha\lambda^\alpha - \bar{\lambda}_\alpha\mathcal{F}^\alpha + \Delta H \tag{205}$$

$$\Delta H = \wp_a V^a_b\eta^b + i\Big\{\wp_a\{\chi^a, G_b\}\eta^b + \wp_c\chi^a U^c_{ab}\eta^b - \wp_a\wp_b\{\chi^a, U^b_{cd}\}\eta^c\eta^d\Big\} \tag{206}$$

Note that **in general four-ghost interaction is present**.

For the usual Yang-Mills theory there are no such terms since the "structure constants" U^b_{cd} are indeed constant and $\{\chi^a, U^b_{cd}\} = 0$.

4.3.2 Proof of the BV Theorem

For simplicity, we will deal only with the case where **the constraints are all bosonic** and hence η^a **are all fermionic.**

To prove the BV theorem, we first derive some crucial identities. We will use the following notations:

$$G = G_a \eta^a \,, \qquad V^a = V_b^a \eta^b \,, \tag{207}$$
$$U_b^a = U_{bc}^a \eta^c \,, \qquad U^a = \frac{1}{2} U_{bc}^a \eta^b \eta^c \tag{208}$$

• They are very similar to differential forms, with η^a playing the role of dx^μ.

• Note that the b, c indices in U^a are contracted oppositely to the ones in the definition of Ω. (This leads to the minus sign below in Ω.)

Then we have

$$\begin{aligned} \Omega &= G - \wp_a U^a \,, & (209) \\ H_\Psi &= H_0 + \wp_a V^a + i\,\{\Psi, \Omega\} \,. & (210) \end{aligned}$$

□ Representation of the gauge algebra:

The first set of identities, which follow directly from the definitions above and the constraint algebra, are

$$\begin{aligned} &(1) \qquad \{G_a, G\} = G_b U^b_a \,, & (211) \\ &(2) \qquad \{G, G\} = 2G_a U^a \,, & (212) \\ &(3) \qquad \{H_0, G\} = G_a V^a \,. & (213) \end{aligned}$$

□ First-level Jacobi identities:

Now we prove what we call the **first-level Jacobi identities**:

$$(\boldsymbol{Ia}) \qquad \{G, U^a\} = U^a_b U^b\,, \tag{214}$$

$$(\boldsymbol{Ib}) \qquad \{G, V^a\} = V^a_b U^b - U^a_b V^b + \{H_0, U^a\} \tag{215}$$

These are called the first-level because they follow from the Jacobi identity for the Poisson brackets involving the constraints and H_0 alone.

Proof of (Ia):

Consider the double Poisson bracket $\{\{G, G\}, G\}$. **Since G is fermionic**, the graded Jacobi identity tells us

$$\begin{aligned}\{\{G, G\}, G\} &= \{G, \{G, G\}\} - \{\{G, G\}, G\} \\ &= \{\{G, G\}, G\} - \{\{G, G\}, G\} = 0\end{aligned} \tag{216}$$

On the other hand, using $(\mathbf{2})$ we get

$$\begin{aligned} \{\{G, G\}, G\} &= 2\{G_a U^a, G\} \\ &= 2\{G_a, G\} U^a + 2G_a \{U^a, G\} \\ &= 2G_a \left(U^a_b U^b + \{U^a, G\}\right) . \end{aligned} \tag{217}$$

Since G_a's are assumed to be algebraically independent, for this to vanish we must have the identity $(\boldsymbol{Ia})$. //

Proof of $(\boldsymbol{Ib})$:

Similarly consider $\{G, \{H_0, G\}\}$ and compute this in two ways. One way is to use the Jacobi identity for the Poisson brackets. The other way is to compute it directly using $(\mathbf{3})$. Equating them we easily get $(\boldsymbol{Ib})$.

Excercise: Show this explilcitly.

□ Second-level Jacobi identities:

Next, we will prove the **second-level Jacobi identities**:

$$(\boldsymbol{IIa}) \qquad \{\boldsymbol{U^a}, \boldsymbol{U^b}\} = \boldsymbol{0}\,, \tag{218}$$

$$(\boldsymbol{IIb}) \qquad \{\boldsymbol{U^a}, \boldsymbol{V^b}\} = \boldsymbol{0}\,. \tag{219}$$

These are called the second-level because **they are relations between the coefficients of the constraint algebra** and are derived using the Poisson bracket Jacobi identities with $\boldsymbol{U}$ or $\boldsymbol{V}$ in one of the slots and the first-level Jacobi identities.

Proof of $(\boldsymbol{IIa})$:

We consider the double Poisson bracket $\{\{\boldsymbol{G}, \boldsymbol{G}\}\,, \boldsymbol{U^a}\}$. By using the Jacobi identity (note $\boldsymbol{U^a}$ is bosonic) we can write this as

$$\begin{aligned} \{\{G, G\}, U^a\} &= \{G, \{G, U^a\}\} + \{\{G, U^a\}, G\} \\ &= 2\{G, \{G, U^a\}\} \\ &= 2\left\{G, U^a_b\right\} U^b - 2U^a_b \left\{G, U^b\right\} , \end{aligned} \tag{220}$$

where in the last line we used $(\boldsymbol{Ia})$.

On the other hand, using (2) we get

$$\begin{aligned} \{\{G, G\}, U^a\} &= 2\left\{G_b U^b, U^a\right\} \\ &= 2G_b \left\{U^b, U^a\right\} + 2\{G_b, U^a\} U^b . \end{aligned} \tag{221}$$

Equating these, we get

$$(*) \quad \left\{G, U^a_b\right\} U^b - U^a_b \{G, U_b\} = G_b \left\{U^b, U^a\right\} + \{G_b, U^a\} U^b \tag{222}$$

To go further, we consider the Jacobi identity

$$0 = \{\{G, G\}, G_a\} - \{\{G, G_a\}, G\} + \{\{G_a, G\}, G\} \tag{223}$$

After some calculations, this Jacobi identity becomes

$$G_c \left(U^c_{ab} U^b + \{U^c, G_a\} - U^c_b U^b_a + \left\{U^c_a, G\right\} \right) = 0 \,. \tag{224}$$

Using the linear independence of G_c, we obtain

$$U^c_{ab} U^b + \{U^c, G_a\} - U^c_b U^b_a + \left\{U^c_a, G\right\} = 0 \,. \tag{225}$$

Now mulitiply this from right by U^a. Then the first term vanishes due to the antisymmetry of U^c_{ab} in $a \leftrightarrow b$ and the result is

$$\{U^c, G_a\} U^a - U^c_b U^b_a U^a + \left\{U^c_a, G\right\} U^a = 0 \,. \tag{226}$$

Substituting this into $(*)$, we get $\left\{U^a, U^b\right\} = 0$. //

Proof of (IIb):

It is obtained similarly by considering $\{\{G, G\}, V^a\}$.

Excercise: Supply the details.

□ Proof of 3 fundamental relations:

We are now ready to prove the following three important relations:

$$\boldsymbol{(a)} \qquad \{\Omega, \Omega\} = 0\,, \tag{227}$$

$$\boldsymbol{(b)} \qquad \{\{\Psi, \Omega\}\,, \Omega\} = 0\,, \tag{228}$$

$$\boldsymbol{(c)} \qquad \{H_0 + \wp_a V^a, \Omega\} = 0\,. \tag{229}$$

Proofs of these relations are straightforward using the formulas already developed. We only show how (a) is proved as an example.

Proof of (a):

Since $\boldsymbol{\Omega}$ is fermionic, this is a non-trivial relation. Recalling $\boldsymbol{\Omega = G - \wp_a U^a}$, we have

$$
\begin{aligned}
\{\Omega, \Omega\} &= \left\{G - \wp_a U^a, G - \wp_b U^b\right\} \\
&= \{G, G\} - 2\{G, \wp_a U^a\} + \left\{\wp_a U^a, \wp_b U^b\right\}
\end{aligned} \tag{230}
$$

(2) gives

$$
\{G, G\} = 2G_a U^a . \tag{231}
$$

Using $(\boldsymbol{Ia})$, the second term of (230) becomes

$$
\begin{aligned}
-2\{G, \wp_a U^a\} &= -2\{G, \wp_a\} U^a + 2\wp_a \{G, U^a\} \\
&= -2G_a U^a + 2\wp_a U_b^a U^b .
\end{aligned} \tag{232}
$$

As for the third term of (230), we use $(\boldsymbol{IIa})$ and get

$$
\begin{aligned}
\left\{\wp_a U^a, \wp_b U^b\right\} &= \{\wp_a U^a, \wp_b\} U^b - \wp_b \left\{\wp_a U^a, U^b\right\} \\
&= \wp_a \{U^a, \wp_b\} U^b - \wp_b \left\{\wp_a, U^b\right\} U^a \\
&= 2\wp_a \{U^a, \wp_b\} U^b \\
&= -2\wp_a U_b^a U^b .
\end{aligned} \tag{233}
$$

Adding all the contributions, we get $\{\Omega, \Omega\} = 0$. //

□ **Ψ independence of Z_Ψ** :

Finally we show the Ψ-independence of Z_Ψ to finish the proof of the BV theorem.

Collectively denote all the variables by $\varphi = (q, p, \eta, \wp)$ and **make a change of variables corresponding to the BRST transformation**

$$\varphi \longrightarrow \tilde{\varphi} = \varphi + \{\varphi, \Omega\}\, \mu \tag{234}$$

♦ μ is a time-independent fermionic parameter, which nevertheless may depend functionally on the dynamical variables.

♦ It is easy to show that the action is invariant. First

$$\delta \int dt \Big(p_A \dot{q}^A + \wp_a \dot{\eta}^a \Big) = \int dt \Big(\delta p_A \dot{q}^A - \dot{p}_A \delta q^A + \delta \wp_a \dot{\eta}^a - \dot{\wp}_a \delta \eta^a \Big) , \tag{235}$$

where we used integration by parts. Now insert the change of variables:

$$\delta q^A = \{q^A, \Omega\}\,\mu = \frac{\partial\Omega}{\partial p_A}\mu\,, \tag{236}$$

$$\delta p_A = \{p_A, \Omega\}\,\mu = -\frac{\partial\Omega}{\partial q^A}\mu\,, \tag{237}$$

$$\delta\eta^a = \{\eta^a, \Omega\}\,\mu = (\partial/\partial\wp_a)\Omega\mu \qquad \text{(left derivative)}\,, \tag{238}$$

$$\delta\wp_a = \{\wp_a, \Omega\}\,\mu = (\partial\Omega/\partial\eta^a)\mu \qquad \text{(right derivative)}\,. \tag{239}$$

Assuming no contributions from the boundary, we get

$$\begin{aligned}
&\delta\int dt\left(p_A\dot{q}^A + \wp_a\dot{\eta}^a\right)\\
&= \int dt\left(-\frac{\partial\Omega}{\partial q^A}\mu\dot{q}^A - \dot{p}_A\frac{\partial\Omega}{\partial p_A}\mu + (\partial\Omega/\partial\eta^a)\mu\dot{\eta}^a - \dot{\wp}_a(\partial/\partial\wp_a)\Omega\mu\right)\\
&= -\int dt\left(\dot{q}^A\frac{\partial\Omega}{\partial q^A} + \dot{p}_A\frac{\partial\Omega}{\partial p_A} + (\partial\Omega/\partial\eta^a)\dot{\eta}^a + \dot{\wp}_a(\partial/\partial\wp_a)\Omega\right)\mu\\
&= -\int dt\frac{d\Omega}{dt}\mu = 0
\end{aligned} \tag{240}$$

♦ Next, the Hamiltonian is invariant since $\delta H_\Psi = \{H_\Psi, \Omega\}\,\mu = 0$.

♦ What remains is the transformation property of the functional measure. The Jacobian is given by

$$
\begin{aligned}
d\tilde{\varphi} &= |J| d\varphi\,, && (241)\\
|J| &= \det \frac{\partial \tilde{\varphi}^j}{\partial \varphi^i} \\
&= \det \left(1 + \frac{\partial}{\partial \varphi^i} \left(\{\varphi^j, \Omega\}\,\mu \right) \right)\,. && (242)
\end{aligned}
$$

Since it will suffice to consider small μ, we can expand

$$
\det(1 + A) = \mathrm{e}^{\mathrm{Tr}\,\ln(1+A)} = 1 + \mathrm{Tr}\,A + \cdots\,. \qquad (243)
$$

So we get

$$\begin{aligned}
|J| &= 1 + \frac{\partial}{\partial q^A}\left(\{q^A, \Omega\}\,\mu\right) + \frac{\partial}{\partial p_A}\left(\{p_A, \Omega\}\,\mu\right) + \cdots \\
&= 1 + \frac{\partial}{\partial q^A}\left(\frac{\partial \Omega}{\partial p_A}\mu\right) + \frac{\partial}{\partial p_A}\left(-\frac{\partial \Omega}{\partial q^A}\mu\right) + \cdots \\
&= 1 + \frac{\partial \Omega}{\partial p_A}\frac{\partial \mu}{\partial q^A} - \frac{\partial \Omega}{\partial q^A}\frac{\partial \mu}{\partial p_A} + \cdots \\
&= 1 - \left(\frac{\partial \mu}{\partial q^A}\frac{\partial \Omega}{\partial p_A} - \frac{\partial \mu}{\partial p_A}\frac{\partial \Omega}{\partial q^A}\right) + \cdots \\
&= 1 - \{\mu, \Omega\} \\
&\simeq \mathrm{e}^{-\{\mu,\Omega\}} = \mathrm{e}^{i(i\{\mu,\Omega\})} \,.
\end{aligned} \tag{244}$$

Now let us choose

$$\mu = \int dt\,(\Psi' - \Psi)\ , \tag{245}$$

where **Ψ' is infinitesimally different from Ψ.** Then, we have

$$d\tilde{\varphi} = d\varphi \mathrm{e}^{i\int dt\left(i\left\{\Psi',\Omega\right\} - i\{\Psi,\Omega\}\right)} \,. \tag{246}$$

Inserting it into the functional integral, we prove

$$\boldsymbol{Z_{\Psi'} = Z_{\Psi}} \quad // \tag{247}$$

4.4 Proof of unitarity: Equivalence with canonical functional Integral

Having proved the BV theorem, we now prove the equivalence with the canonical functional integral, $i.e.$ **the functional integral involving only the physical degrees of freedom.**

The proof proceeds in two steps.

□ **First step**:

The first step is to derive a convenient representation of the **physical** path integral.

Let us make a canonical transformation[5] of the form

$$(\boldsymbol{p_i}, \boldsymbol{q^i}) \longrightarrow (\boldsymbol{p_i^*}, \boldsymbol{q^{*i}}) \,, \qquad \boldsymbol{i = 1 \sim n} \tag{248}$$

where the new coordinates $(\boldsymbol{q^{*i}}, \boldsymbol{p_i^*})_{\boldsymbol{i=1\sim n}}$ are split into the

[5]Canonical transformation is the one which does not change the standard form of the symplectic structure. Namely, $\boldsymbol{\omega} = \sum_i \boldsymbol{dp_i} \wedge \boldsymbol{dq_i} = \sum_i \boldsymbol{dp_i^*} \wedge \boldsymbol{dq^{*i}}$.

physical part $(q^{*u}, p^*_u)_{u=1\sim n-m}$ and the unphysical part $(q^{*\alpha}, p^*_\alpha)_{\alpha=1\sim m}$.
Moreover, **we take p^*_α to be the gauge-fixing constraints themselves,** namely

$$\boxed{p^*_\beta = \Theta^\beta\,, \quad \beta = 1 \sim m \qquad (249)}$$

Then, since canonical transformation does not change the Poisson bracket structure, we have (recall T_α are the constraints)

$$\det \frac{\partial T_\alpha}{\partial q^{*\beta}} = \det \left\{T_\alpha, p^*_\beta\right\} = \det \left\{T_\alpha, \Theta^\beta\right\} \neq 0 \tag{250}$$

By assumption, Θ^β are chosen so that this holds at every point in the phase space.
This then guarantees that using the m constraints $T_\alpha(p^*, q^*) = 0$ one can solve for the m variables $q^{*\alpha}$ in terms of the rest.
For example, consider the constraints near the point where $q^{*\alpha}$ are very small.

Then

$$\boldsymbol{0 = T_\alpha(p^*, q^{*u}, q^{*\beta}) = T_\alpha(p^*, q^{*u}, 0) + \frac{\partial T_\alpha}{\partial q^{*\beta}}(p^*, q^{*u}, 0) q^{*\beta}} \tag{251}$$

Since the matrix $\boldsymbol{M_{\alpha\beta} = \frac{\partial T_\alpha}{\partial q^{*\beta}}(p^*, q^{*u}, 0)}$ is invertible, we can solve for $\boldsymbol{q^{*\alpha}}$ as

$$\boldsymbol{q^{*\alpha} = -(M^{-1})^{\alpha\beta} T_\beta(p^*, q^{*u}, 0)} \tag{252}$$

Since this can be continued away from the origin of $\boldsymbol{q^{*\alpha}}$, we conclude that we can always solve $\boldsymbol{q^{*\alpha}}$ in terms of the other variables:

$$\boxed{\boldsymbol{q^{*\alpha} \;=\; q^{*\alpha}(q^{*u}, p_i^*)} \qquad (253)}$$

Now the physical partition function $\boldsymbol{Z^*}$ is given by

$$\begin{aligned} \boldsymbol{Z^*} \;&=\; \boldsymbol{\int \prod (dp_i^* dq^{*i}) \mathrm{e}^{i \int dt (p_i^* \dot{q}^{*i} - H_0)} \prod \delta(p_\beta^*)} \\ &\quad \boldsymbol{\cdot \prod \delta(q^\alpha - q^{*\alpha}(q^{*u}, p_u^*))} \end{aligned} \tag{254}$$

Note that because of the presence of the δ-functions $\prod \delta(p^*_\beta)$, we can drop the dependence on p^*_β in the expression of $q^{*\alpha}$, so that $p^*_i \longrightarrow p^*_u$.

Finally, we will make use of the following :

- The Liouville measure is invariant. That is $\prod(dp^*_i dq^{*i}) = \prod(dp_i dq^i)$.

- The last δ-function can be rewritten as

$$\begin{aligned} &\prod \delta(q^{*\alpha} - q^{*\alpha}(q^{*u}, p^*_u)) \prod \delta(p^*_\beta) \\ &= \prod \delta(T_\alpha) \det \left(\frac{\partial T_\alpha}{\partial q^{*\beta}}\right) \prod \delta(p^*_\beta) \\ &= \prod \delta(T_\alpha) \det \left\{T_\alpha, \Theta^\beta\right\} \prod \delta(\Theta^\beta) \end{aligned} \tag{255}$$

We then finally obtain a useful form of the physical partition function

$$\boxed{Z^* = \int \prod (dp_i dq^i) \prod \delta(T_\alpha) \prod \delta(\Theta^\beta) \det \left\{T_\alpha, \Theta^\beta\right\} e^{i \int dt (p_i \dot{q}^i - H_0)}}$$

Clearly this form is quite natural.

□ **Second Step**:

We now want to prove that the above form is reproduced from the BV theorem. In the BV theorem, let us take the gauge fixing function to be of the form

$$\Theta^\alpha = \dot{\lambda}^\alpha + \mathcal{F}^\alpha(q, p) \tag{256}$$

It is convenient to divide the ghosts as

$$\eta^a = (\tilde{\eta}^\alpha, \eta^\alpha) , \qquad \wp_a = (\tilde{\wp}_\alpha, \wp_\alpha) \tag{257}$$

Then the BV action takes the form

$$S_\Psi = \int dt\left(p_i\dot{q}^i + \bar{\lambda}_\alpha\dot{\lambda}^\alpha + \wp_\alpha\dot{\eta}^\alpha + \tilde{\wp}_\alpha\dot{\tilde{\eta}}^\alpha - H_\Psi\right) \tag{258}$$

where

$$H_\Psi = H_0 - T_\alpha\lambda^\alpha - \bar{\lambda}_\alpha\mathcal{F}^\alpha + \Delta H \tag{259}$$

$$\Delta H = \wp_a V_b^a\eta^b + i\Big[\wp_a\{\chi^a, G_b\}\eta^b + \wp_c\chi^a U_{ab}^c\eta^b - \wp_a\wp_b\{\chi^a, U_{cd}^b\}\eta^c\eta^d\Big] \tag{260}$$

Recalling the extended definitions of G_a, χ^a (as in (188) and (189)), V_b^a and U_{ab}^c, we can write each term in ΔH more explicitly:

$$\wp_a V_b^a\eta^b = \wp_\alpha V_\beta^\alpha\eta^\beta \tag{261}$$

$$\wp_a\{\chi^a, G_b\}\eta^b = -\wp_\alpha\tilde{\eta}^\alpha + \tilde{\wp}_\gamma\{\mathcal{F}^\gamma, T_\beta\}\eta^\beta \tag{262}$$

$$\wp_c\chi^a U_{ab}^c\eta^b = \wp_\gamma i\lambda^\alpha U_{\alpha\beta}^\gamma\eta^\beta \tag{263}$$

$$\wp_a\wp_b\{\chi^a, U_{cd}^b\}\eta^c\eta^d = \tilde{\wp}_\alpha\wp_\beta\left\{\mathcal{F}^\alpha, U_{\gamma\delta}^\beta\right\}\eta^\gamma\eta^\delta \tag{264}$$

We now make the following **rescaling by a global parameter ϵ**, which will be taken to zero:

$$(i) \qquad \mathcal{F}^{\alpha} \longrightarrow \frac{1}{\epsilon}\mathcal{F}^{\alpha} \tag{265}$$

$$(ii) \qquad \bar{\lambda}_{\alpha} \longrightarrow \epsilon\bar{\lambda}_{\alpha} \tag{266}$$

$$(iii) \qquad \tilde{\wp}_{\alpha} \longrightarrow \epsilon\tilde{\wp}_{\alpha} \tag{267}$$

• The functional measure is invariant since (ii) and (iii) are bosonic and fermionic and the factors of ϵ cancel.

• **The purpose of this rescaling is to get rid of the terms $\bar{\lambda}_{\alpha}\dot{\lambda}^{\alpha}$ and $\tilde{\wp}_{\alpha}\dot{\tilde{\eta}}^{\alpha}$**, which scale like ϵ and vanish as $\epsilon \longrightarrow 0$. All the other terms in the action remain unchanged.

• Now since $\dot{\tilde{\eta}}^{\alpha}$ has dissappeared, $\tilde{\eta}^{\alpha}$ only appears as $-\wp_{\alpha}\tilde{\eta}^{\alpha}$ in (262). Thus upon integration over $\tilde{\eta}^{\alpha}$, we get $\delta(\wp_{\alpha})$.

This in turn allows us to integrate out $\wp_\alpha$ and the action simplifies drastically to

$$S_\Psi = \int dt \left(p_i \dot{q}^i - H_0 + T_\alpha \lambda^\alpha + \bar{\lambda}_\alpha \mathcal{F}^\alpha - i\tilde{\wp}_\alpha \left\{ \mathcal{F}^\alpha, T_\beta \right\} \eta^\beta \right) \tag{268}$$

Now **upon integration over $\tilde{\wp}_\alpha$ and η^β, we get det $\left\{ T_\alpha, \mathcal{F}^\beta \right\}$**. **Thus we precisely get the canonical path integral with the gauge-fixing function Θ^α replaced by $\mathcal{F}^\alpha(q, p)$.** *q.e.d.*

Remark: In the application to non-abelian gauge theory, to retain the relativistic form, we will not perform the rescaling and simply integrate out $\wp_\alpha$. The details will be discussed later.

5 Application to non-abelian gauge theory

In this chapter, we apply the methods developed previously to the important case of non-abelian gauge theory.

5.1 Dirac's Method

First, we make use of the Dirac's method. The basic procedure is the same as the abelian case already described but the details are more involved.

For simplicity, we take the gauge group G to be **compact** and normalize the group metric to be of the form $g_{ab} = \delta_{ab}$. Thus as far as the group indices are concerned, we need not distinguish upper and lower ones. Where convenient, we will use the notations

$$A \cdot B \equiv A^a B_a \,, \tag{269}$$

$$(A \times B)_a \equiv f_{abc} A^b B^c \,. \tag{270}$$

Due to the complete antisymmetry of the structure constant, we have the cyclic identity

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \,. \tag{271}$$

□ Conjugate momenta and the basic Poisson bracket:

The Lagrangian is given by

$$\boldsymbol{L} = -\frac{1}{4}\int d^3x\, \boldsymbol{F}_{\mu\nu}\cdot \boldsymbol{F}^{\mu\nu}\,, \tag{272}$$

$$\boldsymbol{F}_{\mu\nu} = \boldsymbol{A}_{\nu,\mu} - \boldsymbol{A}_{\mu,\nu} + g\boldsymbol{A}_\mu \times \boldsymbol{A}_\nu\,. \tag{273}$$

We will regard A^a_μ as our fundamental variables. The conjugate momenta are found by the variation with respect to $\dot{A}^a_\mu = A^a_{\mu,0}$:

$$\begin{aligned} \delta \boldsymbol{L} &= -\tfrac{1}{2}\int d^3x\, \boldsymbol{F}_{\mu\nu}\cdot \delta \boldsymbol{F}^{\mu\nu} \\ &= +\int d^3x\, \boldsymbol{F}^{\mu 0}\delta \boldsymbol{A}_{\mu,0} \end{aligned}$$

$$\therefore \qquad \Pi^\mu_a = F^{\mu 0}_a\,. \tag{274}$$

The equal time (ET) Poisson bracket is defined by

$$\left\{A^a_\mu(x), \Pi^\nu_b(y)\right\} = \delta^a_b \delta^\nu_\mu \delta(\vec{x}-\vec{y})\,. \tag{275}$$

From (274) we get **the primary constraints:**

$$\Pi^0_a = F^{00}_a = 0\,. \tag{276}$$

□ Canonical Hamiltonian:

The canonical Hamiltonian is given by

$$\begin{aligned} \boldsymbol{H_{can}} &= \int \boldsymbol{d^3x\, \Pi^\mu \cdot A_{\mu,0} - L} \\ &= \int \boldsymbol{d^3x} \left(\boldsymbol{F^{i0} \cdot A_{i,0} + \frac{1}{4} F^{ij} \cdot F_{ij} + \tfrac{1}{2} F^{i0} \cdot F_{i0}} \right) . \end{aligned} \tag{277}$$

Rewriting the first term as

$$\begin{aligned} \boldsymbol{F^{i0} \cdot A_{i,0}} &= \boldsymbol{F^{i0} \cdot (A_{i,0} - A_{0,i} + gA_0 \times A_i) + F^{i0} \cdot A_{0,i} - gF^{i0} \cdot (A_0 \times A_i)} \\ &= \boldsymbol{-F^{i0} \cdot F_{i0} + F^{i0} \cdot A_{0,i} - gF^{i0} \cdot (A_0 \times A_i)} \,, \end{aligned} \tag{278}$$

we get

$$\begin{aligned} \boldsymbol{H_{can}} &= \int \boldsymbol{d^3x} \left(\boldsymbol{\frac{1}{4} F^{ij} \cdot F_{ij} - \tfrac{1}{2} F^{i0} \cdot F_{i0} + F^{i0} \cdot A_{0,i} - gF^{i0} \cdot (A_0 \times A_i)} \right) \\ &= \int \boldsymbol{d^3x} \left(\boldsymbol{\frac{1}{4} F^{ij} \cdot F_{ij} + \tfrac{1}{2} \Pi^i \cdot \Pi^i - A_0 \cdot} \left(\underbrace{\boldsymbol{\partial_i \Pi^i + gA_i \times \Pi^i}}_{\boldsymbol{D_i \Pi^i}} \right) \right) \end{aligned} \tag{279}$$

In the last line we used integration by parts and the cyclic identity. The expression in the parenthesis is identified as the covariant derivative:

$$\boldsymbol{D_i \Pi^i \equiv \partial_i \Pi^i + g A_i \times \Pi^i} \,. \tag{280}$$

Consistency between $\boldsymbol{H_{can}}$ and the primary constraint $\boldsymbol{\Pi_a^0 \sim 0}$ immediately gives **the secondary constraint**

$$\boxed{\boldsymbol{\mathcal{G} \equiv D_i \Pi^i = 0} \qquad \text{(Gauss' Law constraint)} \tag{281}}$$

Note that $\boldsymbol{\mathcal{G}}$ does not contain $\boldsymbol{\Pi^0}$ nor $\boldsymbol{A_0}$.

□ **Consistency with $\boldsymbol{H_{can}}$ and the algebra of constraints**:

We must study the compatibility with $\boldsymbol{H_{can}}$ and the algebra of constraints. For this purpose, let us develop some identities. First from the basic Poisson bracket, we have (all the bracktes are at equal time)

$$\left\{A_0^a(x), \mathcal{G}_b(y)\right\} = 0\,, \tag{282}$$

$$\left\{\Pi_a^0(x), \mathcal{G}_b(y)\right\} = 0\,, \tag{283}$$

$$\left\{A_i^a(x), \mathcal{G}_b(y)\right\} = \frac{\delta}{\delta \Pi_a^i(x)} \mathcal{G}_b(y)\,, \tag{284}$$

$$\left\{\Pi_a^i(x), \mathcal{G}_b(y)\right\} = -\frac{\delta}{\delta A_i^a(x)} \mathcal{G}_b(y)\,. \tag{285}$$

So to compute them, we simply vary $\mathcal{G}$:

$$\delta \mathcal{G} = \partial_i \delta \Pi^i + g \delta A_i \times \Pi^i + g A_i \times \delta \Pi^i\,. \tag{286}$$

From this it follows

$$\left\{A_i^a(x), \mathcal{G}_b(y)\right\} = (D_i^y)_{ab}\, \delta(\vec{x} - \vec{y})\,, \tag{287}$$

$$\text{where} \qquad (D_i^y)_{ab} = \delta_{ab} \partial_i^y + g f_{abc} A_i^c(y)\,, \tag{288}$$

$$\left\{\Pi_a^i(x), \mathcal{G}_b(y)\right\} = g f_{abc} \Pi_c^i \delta(\vec{x} - \vec{y})\,. \tag{289}$$

For a general local function $\boldsymbol{F}\left[\boldsymbol{A}_{\mu}, \boldsymbol{\Pi}^{\mu}\right]$ we have

$$\begin{aligned}\{\boldsymbol{F}(\boldsymbol{x}), \boldsymbol{\mathcal{G}}_b(\boldsymbol{y})\} \; = \; & \int \boldsymbol{d}^3\boldsymbol{z}\left(\frac{\boldsymbol{\delta F}(\boldsymbol{x})}{\boldsymbol{\delta A}_i^c(\boldsymbol{z})}\left\{\boldsymbol{A}_i^c(\boldsymbol{z}), \boldsymbol{\mathcal{G}}_b(\boldsymbol{y})\right\}\right. \\ & \left. + \frac{\boldsymbol{\delta F}(\boldsymbol{x})}{\boldsymbol{\delta \Pi}_c^i(\boldsymbol{z})}\left\{\boldsymbol{\Pi}_c^i(\boldsymbol{z}), \boldsymbol{\mathcal{G}}_b(\boldsymbol{y})\right\}\right). \qquad (290)\end{aligned}$$

Thus we get

$$\begin{aligned}
\{\mathcal{G}_a(x), \mathcal{G}_b(y)\} &= \int d^3z \left(\frac{\delta \mathcal{G}_a(x)}{\delta A_i^c(z)} \frac{\delta \mathcal{G}_b(y)}{\delta \Pi_c^i(z)} - \frac{\delta \mathcal{G}_b(y)}{\delta A_i^c(z)} \frac{\delta \mathcal{G}_a(x)}{\delta \Pi_c^i(z)} \right) \\
&= \int d^3z \left\{ \left(-g f_{cap} \Pi_p^i \delta(\vec{x} - \vec{z}) \right) \cdot (D_i^y)_{cb}\, \delta(\vec{z} - \vec{y}) \right. \\
&\qquad \left. - ((a,x) \leftrightarrow (b,y)) \right\} \\
&= \int d^3z \left\{ -g f_{cap} \Pi_p^i(x) \left(\partial_i^y \delta_{cb} + g f_{cbq} A_i^q(y) \right) \delta(\vec{x} - \vec{y}) \right. \\
&\qquad \left. - ((a,x) \leftrightarrow (b,y)) \right\} \\
&= g f_{abc} \Pi_c^i(x) \partial_i^y \delta(\vec{x} - \vec{y}) \\
&\qquad - g^2 f_{cap} f_{cbq} \Pi_p^i(x) A_i^q(x) \delta(\vec{x} - \vec{y}) - ((a,x) \leftrightarrow (b,y)) \ .
\end{aligned}$$

When the terms obtained by $(a,x) \leftrightarrow (b,y)$ is explicitly taken into account, the first term of $\mathcal{O}(g)$ becomes, omitting gf_{abc},

$$\begin{aligned}
&\Pi^i_c(x)\partial^y_i\delta(\vec{x}-\vec{y})+\Pi^i_c(y)\partial^x_i\delta(\vec{x}-\vec{y}) \\
&\quad=\partial^y_i\left(\Pi^i_c(y)\delta(\vec{x}-\vec{y})\right)-\Pi^i_c(y)\partial^y_i\delta(\vec{x}-\vec{y}) \\
&\quad=\partial^x_i\Pi^i_c(x)\delta(\vec{x}-\vec{y})\,. \qquad (291)
\end{aligned}$$

As for the $\mathcal{O}(g^2)$ term, use the Jacobi identity for the structure constant:

$$f_{cap}f_{cbq}-f_{cbp}f_{caq} = f_{abc}f_{cpq}\,. \qquad (292)$$

Therefore, we get

$$\begin{aligned}
\{\mathcal{G}_a(x),\mathcal{G}_b(x)\} &= gf_{abc}\Big(\partial^x_i\Pi^i_c(x)-gf_{cpq}\Pi^i_p(x)A^q_i(x)\Big)\,\delta(\vec{x}-\vec{y}) \\
&= gf_{abc}\mathcal{G}_c(x)\delta(\vec{x}-\vec{y})\,. \qquad (293)
\end{aligned}$$

Therefore, $\mathcal{G}_a(x)$**'s form a closed local gauge algebra**. (Note there is no i classically.)

Next compute $\{\boldsymbol{H_{can}}, \mathcal{G}_b(\boldsymbol{y})\}$. It is given by

$$\{\boldsymbol{H_{can}}, \mathcal{G}_b(\boldsymbol{y})\} = \boldsymbol{P_1} + \boldsymbol{P_2}\,, \tag{294}$$

$$\boldsymbol{P_1} = \int \boldsymbol{d^3x} \left\{ \frac{1}{4} \boldsymbol{F^{jk} \cdot F_{jk}(x)} + \tfrac{1}{2} \boldsymbol{\Pi^i \cdot \Pi^i(x)}, \mathcal{G}_b(\boldsymbol{y}) \right\}\,, \tag{295}$$

$$\boldsymbol{P_2} = -\int \boldsymbol{d^3x}\, \boldsymbol{A_0^a} \left\{ \mathcal{G}_a(\boldsymbol{x}), \mathcal{G}_b(\boldsymbol{y}) \right\}\,. \tag{296}$$

$\boldsymbol{P_2}$ has already been computed. So we concentrate on $\boldsymbol{P_1}$. From the definition of Poisson bracket, we have

$$
\begin{aligned}
\boldsymbol{P_1} &= \int d^3x \int d^3z \left\{ \frac{\delta}{\delta A_i^c(z)} \left(\frac{1}{4} F^{jk} \cdot F_{jk}(x) \right) \frac{\delta \mathcal{G}_b(y)}{\delta \Pi_c^i(z)} \right. \\
&\qquad \left. - \frac{\delta \mathcal{G}_b(y)}{\delta A_i^c(z)} \frac{\delta}{\delta \Pi_c^i(z)} \left(\tfrac{1}{2} \Pi^i \cdot \Pi^i(x) \right) \right\} \\
&= \int d^3x \int d^3z \left\{ \tfrac{1}{2} F_a^{jk}(x) \frac{\delta F_{jk}^a(x)}{\delta A_i^c(z)} \left(D_i^y \right)_{cb} \delta(\vec{z} - \vec{y}) \right. \\
&\qquad \left. + g f_{cbp} \Pi_p^i(y) \delta(\vec{z} - \vec{y}) \Pi_c^i(x) \delta(\vec{x} - \vec{y}) \right\}. \qquad (297)
\end{aligned}
$$

The last term vanishes due to the antisymmetry of the structure constant. Performing the functional differentiation, we get

$$
\begin{aligned}
\boldsymbol{P_1} &= \int d^3x \int d^3z \left\{ \tfrac{1}{2} F^{ji} \partial_j^x \delta(\vec{x} - \vec{z}) - \tfrac{1}{2} F_c^{ij} \partial_j^x \delta(\vec{x} - \vec{z}) \right. \\
&\qquad \left. + \tfrac{1}{2} F_z^{ik} g f_{acq} A_k^q \delta(\vec{x} - \vec{z}) + \tfrac{1}{2} F_a^{ji} g f_{apc} A_j^p \delta(\vec{x} - \vec{z}) \right\} \left(D_i^y \right)_{cb} \delta(\vec{z} - \vec{y}).
\end{aligned}
$$

The first two terms in the parenthesis are identical. The next two terms are also the same. The latter times the covariant derivative term, after z-integration, becomes

$$\begin{aligned} & g\int d^3x\, F_a^{ik} A_k^q f_{acq}\left(\partial_i^y \delta_{cb} + g f_{cbp} A_i^p\right)\delta(\vec{x}-\vec{y}) \\ & = g f_{abc}\int d^3x\,\partial_i^y\left(F_a^{ik}A_k^c\right)(y)\delta(\vec{x}-\vec{y}) \\ & \quad -g^2\int d^3x\, f_{acq}f_{bcp}F_a^{ik}A_k^q A_i^p(y)\delta(\vec{x}-\vec{y})\,. \end{aligned} \tag{298}$$

For the last term, because of antisymmetry of F_a^{ik}, we can antisymmetrize with respect to $q \leftrightarrow p$ and then use the Jacobi identity

$$-f_{acq}f_{bcp} + f_{acp}f_{bcq} \,=\, f_{abc}f_{cpq}\,. \tag{299}$$

Factoring out gf_{abc}, the terms with the common factor F_a^{ik} combine together to form F_{ik}^c and we get a contribution to P_1 of the form

$$g f_{abc}\int d^3x\,\left(\partial_i F_a^{ik}A_k^c + \tfrac{1}{2}F_a^{ik}F_{ik}^c\right)\delta(\vec{x}-\vec{y}) \,=\, g f_{abc}\partial_i F_a^{ik}A_k^c \tag{300}$$

where the last term on the LHS vanishes due to antisymmetry of f_{abc}.

The remaining contribution to P_1 reads

$$
\begin{aligned}
-\int d^3x\, \partial_j F_c^{ji}(x)\, (D_i^y)_{cb}\, \delta(\vec{x}-\vec{y}) \\
= -\int d^3x\, \partial_j F_b^{ji}(x) \partial_i^y \delta(\vec{x}-\vec{y}) \\
-\int d^3x\, \partial_j F_c^{ji}(x) g f_{cbp} A_i^p \delta(\vec{x}-\vec{y}) \\
= -g f_{abc} \partial_j F_a^{ji} A_i^c(y)\,.
\end{aligned} \tag{301}
$$

This precisely cancels the contribution already computed and we thus get $P_1 = 0$.

So only P_2 remains and we finally obtain

$$
\boxed{\{H_{can}, \mathcal{G}_b(y)\} = -g f_{abc} A_0^a \mathcal{G}_c(y) \sim 0\,.} \tag{302}
$$

This shows that **no new constraints are generated.**

□ Gauge-fixing and the Dirac brackets:

Summarizing, we have found the following constraints:

$$\phi_a(x) \equiv \Pi^0_a(x) = 0 \tag{303}$$

$$\chi_a(x) \equiv \mathcal{G}_a(x) = (D_i\Pi^i)_a(x) = 0 \tag{304}$$

We may take the Coulomb gauge as in the abelian case by imposing the additional constraints:

$$\tilde{\phi}_a(x) \equiv A_{0,a}(x) = 0 \tag{305}$$

$$\tilde{\chi}_a(x) \equiv \partial_i A^i_a(x) = 0 \tag{306}$$

Excercise: Compute the Dirac brackets among the fundamental variables A^i_a and Π^a_j.

5.2 Application of the BFV formalism

Next we describe the BFV method for the non-abelian gauge theory in some detail.

5.2.1 Recollection of some notations and results

We will take the gauge-fixing constraint to be of the form

$$\boxed{\boldsymbol{\Theta}^{\alpha} = \dot{\boldsymbol{\lambda}}^{\alpha} + \boldsymbol{\mathcal{F}}^{\alpha} \qquad (307)}$$

As before the following notations will be employed:

$$\boldsymbol{q}^{A} = (\boldsymbol{q}^{i}, \boldsymbol{\lambda}^{\alpha}), \qquad \boldsymbol{p}_{A} = (\boldsymbol{p}_{i}, \bar{\boldsymbol{\lambda}}_{\alpha}), \tag{308}$$

$$\boldsymbol{G}_{a} = (\boldsymbol{T}_{\alpha}, i\bar{\boldsymbol{\lambda}}_{\alpha}), \qquad \boldsymbol{\chi}^{a} = (i\boldsymbol{\lambda}^{\alpha}, \boldsymbol{\mathcal{F}}^{\alpha}), \tag{309}$$

$$\boldsymbol{\eta}^{a} = (\boldsymbol{\eta}^{\alpha}, \tilde{\boldsymbol{\eta}}^{\alpha}), \qquad \boldsymbol{\wp}_{a} = (\boldsymbol{\wp}_{\alpha}, \tilde{\boldsymbol{\wp}}_{\alpha}). \tag{310}$$

The **gauge-fermion** Ψ is taken to be of the form

$$\boldsymbol{\Psi = \wp_a \chi^a} \tag{311}$$

So the **gauge-fixing term** in the action $\boldsymbol{S_\Psi}$ is $\boldsymbol{-i\{\Psi, \Omega\}}$ where

$$\begin{aligned} \boldsymbol{\{\Psi, \Omega\}} &= \boldsymbol{\left\{\wp_a \chi^a, G_b \eta^b - \tfrac{1}{2} \wp_b U^b_{cd} \eta^c \eta^d\right\}} \\ &= \boldsymbol{\wp_a \{\chi^a, G_b\} \eta^b + \chi^a G_a} \\ &\qquad \boldsymbol{- \wp_a \wp_b \left\{\chi^a, U^b_{cd}\right\} \eta^c \eta^d + \wp_c \chi^a U^c_{ab} \eta^b} \end{aligned} \tag{312}$$

5.2.2 Assumptions relevant for gauge theories

We will consider the following case, which is relevant for non-abelian gauge theories.

$$\begin{aligned} \left\{\chi^a, U^b_{cd}\right\} &= 0\,, && (313)\\ \{\lambda^\alpha, T_\beta\} &= 0\,, && (314)\\ \left\{\mathcal{F}^\alpha, \bar{\lambda}_\beta\right\} &= 0\,, && (315)\\ \{T_\alpha, T_\beta\} &= T_\gamma U^\gamma_{\alpha\beta}\,, \qquad \text{Rest} = 0 && (316) \end{aligned}$$

In this case the BRST operator and the gauge-fixing term simplify as follows:

$$\Omega = T_\alpha \eta^\alpha + i\bar{\lambda}_\alpha \tilde{\eta}^\alpha - \tfrac{1}{2}\wp_\alpha U^\alpha_{\beta\gamma}\eta^\beta\eta^\gamma \tag{317}$$

$$\begin{aligned} \{\Psi, \Omega\} &= \wp_\alpha \left\{i\lambda^\alpha, i\bar{\lambda}_\beta\right\} \tilde{\eta}^\beta + \tilde{\wp}_\alpha \{\mathcal{F}^\alpha, T_\beta\}\, \eta^\beta \\ &\qquad + i\wp_\gamma \lambda^\alpha U^\gamma_{\alpha\beta}\eta^\beta + i\lambda^\alpha T_\alpha + i\mathcal{F}^\alpha \bar{\lambda}_\alpha\,, \\ &= -\wp_\alpha \tilde{\eta}^\alpha + \tilde{\wp}_\alpha \{\mathcal{F}^\alpha, T_\beta\}\, \eta^\beta \\ &\qquad + i\wp_\gamma \lambda^\alpha U^\gamma_{\alpha\beta}\eta^\beta + i\lambda^\alpha T_\alpha + i\mathcal{F}^\alpha \bar{\lambda}_\alpha \end{aligned} \tag{318}$$

- Note that there are terms in which different types of ghosts, with and without tilde, are mixed.

□ BRST transformation of the fields:

It is easy to compute the BRST transformations of the fields:

$$
\begin{aligned}
\Omega &= T_\alpha \eta^\alpha + i\bar{\lambda}_\alpha \tilde{\eta}^\alpha - \tfrac{1}{2}\wp_\alpha U^\alpha_{\beta\gamma}\eta^\beta\eta^\gamma \\
\{\wp_\alpha, \Omega\} &= T_\alpha + \wp_\beta U^\beta_{\alpha\gamma}\eta^\gamma \\
\{\tilde{\wp}_\alpha, \Omega\} &= i\bar{\lambda}_\alpha\,, \quad \left\{\bar{\lambda}, \Omega\right\} = 0 \qquad \text{doublet} \\
\{\eta_\alpha, \Omega\} &= -\tfrac{1}{2}U^\alpha_{\beta\gamma}\eta^\beta\eta^\gamma \\
\left\{q^i, \Omega\right\} &= \left\{q^i, T_\alpha\right\}\eta^\alpha - \tfrac{1}{2}\wp_\alpha\left\{q^i, U^\alpha_{\beta\gamma}\right\}\eta^\beta\eta^\gamma \\
&\bullet \qquad \text{similarly for } p_i \\
\{\lambda^\alpha, \Omega\} &= i\tilde{\eta}^\alpha\,, \quad \{\tilde{\eta}^\alpha, \Omega\} = 0 \qquad \text{doublet}
\end{aligned}
$$

5.2.3 Form of the Action

With the above expression for the gauge fixing term, the action takes the form

$$S = \int dt \Big(p_i \dot{q}^i + \wp_\alpha \dot{\eta}^\alpha + \underbrace{\tilde{\wp}_\alpha \dot{\tilde{\eta}}^\alpha}_{(*)} - H_0 + \lambda^\alpha T_\alpha + \bar{\lambda}_\alpha \Theta^\alpha$$

$$+ i\wp_\alpha \tilde{\eta}^\alpha \underbrace{- i\tilde{\wp}_\alpha \{\mathcal{F}^\alpha, T_\beta\} \eta^\beta}_{(*)} + \wp_\gamma \lambda^\alpha U^\gamma_{\alpha\beta} \eta^\beta \Big) . \quad (319)$$

- Note that $\mathcal{F}^\alpha \bar{\lambda}_\alpha$ term got combined with $\bar{\lambda}_\alpha \dot{\lambda}^\alpha$ (contained in $p_A \dot{q}^A$) to form precisely $\bar{\lambda}_\alpha \Theta^\alpha$.

5.2.4 Simplification of Ghost Sector

We now simplify the ghost sector by **integrating over $\wp_\alpha$**. Then we get the δ-function

$$\delta(\dot{\eta}^\alpha + i\tilde{\eta}^\alpha + U^\alpha_{\beta\gamma} \lambda^\beta \eta^\gamma) . \quad (320)$$

Solving for $\tilde{\eta}^\alpha$ we get

$$\tilde{\eta}^\alpha = i\left(\dot{\eta}^\alpha + U^\alpha_{\beta\gamma}\lambda^\beta\eta^\gamma\right) \equiv i(D_0(\lambda)\eta)^\alpha \,. \tag{321}$$

Eliminating $\tilde{\eta}^\alpha$ using this expression, the remaining ghost sector ($(*)$ parts in (319)) becomes

$$i\tilde{\wp}_\alpha \partial_t \left(D_0(\lambda)\eta\right)^\alpha - i\tilde{\wp}_\alpha \left\{\mathcal{F}^\alpha, T_\beta\right\}\eta^\beta \,. \tag{322}$$

Note that we now have **a second order system** ($i.e.$ with two derivatives). For convenience, we will **rename the ghost variables**:

$$\boxed{\begin{aligned} \tilde{\wp}_\alpha &\longrightarrow \bar{c}_\alpha && (323) \\ \eta^\alpha &\longrightarrow c^\alpha && (324) \end{aligned}}$$

Action now takes the form suitable for gauge theories:

$$\boldsymbol{S = \int dt \Big(p_i \dot{q}^i - H_0 + \lambda^\alpha T_\alpha + \bar{\lambda}_\alpha \Theta^\alpha} \\ \boldsymbol{- i\dot{\bar{c}}_\alpha \left(D_0(\lambda)c\right)^\alpha - i\bar{c}_\alpha \left\{\mathcal{F}^\alpha, T_\beta\right\} c^\beta \Big)} \tag{325}$$

□ **A remark on this form of the action**:

It is instructive to compute the Hamiltonian from this action.

The conjugate momenta are (using right-derivatives)

$$\Pi_{\lambda^\alpha} = \bar{\lambda}_\alpha \tag{326}$$

$$\Pi_i = p_i \tag{327}$$

$$\Pi_{c^\alpha} = -i\dot{\bar{c}}_\alpha \tag{328}$$

$$\Pi_{\bar{c}_\alpha} = i(D_0(\lambda)c)^\alpha = i(\dot{c}^\alpha + U^\alpha_{\beta\gamma}\lambda^\beta c^\gamma) \tag{329}$$

Then the Hamiltonian is easily computed as

$$\begin{aligned} H = H_0 - \lambda_\alpha T^\alpha - \bar{\lambda}_\alpha \mathcal{F}^\alpha + \Pi_{c^\alpha}\left(\frac{1}{i}\Pi_{\bar{c}_\alpha} - U^\alpha_{\beta\gamma}\lambda^\beta c^\gamma\right) \\ - i\bar{c}_\alpha \left[\mathcal{F}^\alpha, T_\beta\right] c^\beta \end{aligned} \tag{330}$$

If we go back to the original notation, such as $c^\alpha \longrightarrow \eta^\alpha$ etc., we notice that this is precisely the one given by (319). In this sense, as far as the ghost sector is concerned, (325) can be considered as the **the Lagrangian form.**

This of course is not surprising: One can go from the Hamiltonian to the Lagrangian formulation by integratin gover the momentum.

5.2.5 The case of Feynman gauge

To obtain the ususal Feynman gauge action, we take $\mathcal{F}^{\alpha}$ to be of the form

$$\begin{aligned} \mathcal{F}^{\alpha} &= (\mathcal{F}^{a}, \mathcal{F}_0^{a}) \,, & (331) \\ \mathcal{F}^{a} &= \partial_i A_a^i + \frac{\alpha}{2}\bar{\lambda}_a \,, & (332) \\ \mathcal{F}_0^{a} &= m A_0^a \,, & (333) \end{aligned}$$

Here and hereafter, the index "a" represents the gauge index.

Also we introduced an arbitrary mass scale m so that the dimensions of $\mathcal{F}^{a}$ and $\mathcal{F}_0^{a}$ agree.

As for T_{α} and the ghosts, we have

$$\begin{aligned} T_{\alpha} &= (\mathcal{G}_a, m\Pi_a^0) \,, & (334) \\ c^{\alpha} &= (c^a, c_0^a) \,. & (335) \end{aligned}$$

Again, we introduced a mass scale m for dimensional purpose.

□ Calculation of $\{\mathcal{F}^\alpha, \mathcal{G}_\beta\}$:

Let us compute the components of $\{\mathcal{F}^\alpha, \mathcal{G}_\beta\}$.

First

$$\begin{aligned} \{\mathcal{F}^a(x), \mathcal{G}_b(y)\} &= \left\{\partial_i A^i_a(x), \mathcal{G}_b(y)\right\} \\ &= -\partial^x_i \left\{A^a_i(x), \partial_j \Pi^j_b(y) + g f_{bcd} A^c_j(y) \Pi^j_d(y)\right\} \\ &= -\partial^x_i \partial^i_x \delta^a_b \delta(\vec{x} - \vec{y}) + g f_{abc} A^i_c \partial^x_i \delta(\vec{x} - \vec{y}) \,. \end{aligned} \tag{336}$$

Thus,

$$\begin{aligned} \bar{c}^a \left\{\mathcal{F}^a, \mathcal{G}_b\right\} c^b &= -\int d^3x \bar{c}^a \partial_i \left(\partial^i \delta^a_b - g f_{abc} A^i_c\right) c^b \\ &= -\int d^3x \bar{c}^a \partial_i (D^i c)^a \,. \end{aligned} \tag{337}$$

The other non-vanishing component is

$$\begin{aligned} \left\{\mathcal{F}^a_0(x), \mathcal{G}^0_b(y)\right\} &= m^2 \left\{A^a_0(x), \Pi^0_b(y)\right\} \\ &= m^2 \delta^a_b \delta(\vec{x} - \vec{y}) \,. \end{aligned} \tag{338}$$

Therefore,

$$\bar{c}^0_a \left\{\mathcal{F}^a_0, \mathcal{G}^0_b\right\} c^b_0 = \int d^3x \bar{c}^0_a m^2 c^a_0 \,. \tag{339}$$

□ Integration over some of the momenta:

Putting altogether, the action becomes,

$$\begin{aligned} S = \int d^4x \Big\{ & \Pi^i_a \dot{A}^a_i - H_0(\Pi^i, A_i) + \lambda^a (D_i \Pi^i)^a \\ & + \Pi^0_a (\dot{A}^a_0 + m\lambda^a_0) + \bar{\lambda}^0_a (\dot{\lambda}^a_0 + mA^a_0) \\ & + \bar{\lambda}_a \left(\dot{\lambda}^a + \partial_i A^i_a + \frac{\alpha}{2} \bar{\lambda}_a \right) \\ & + i\bar{c}^0_a (\partial^2_t - m^2) c^a_0 + i\bar{c}_a \left(\partial_0 (D^0(\lambda)^a_b + \partial_i (D^i)^a_b \right) c^b \Big\} \,. \end{aligned} \tag{340}$$

♦ Integration over Π^i_a for the first line gives back the original Lagrangian where

λ^a plays the role of A_0^a:

$$\Pi_a^i \dot{A}_i^a - H_0(\Pi^i, A_i) + \lambda^a (D_i \Pi^i)^a \Longrightarrow \mathcal{L}_0 = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} \,. \tag{341}$$

♦ Integration over Π_a^0 and $\bar{\lambda}_a^0$ gives two δ-functions. Further integration over A_0^a reduces this to

$$\delta\left((\partial_t^2 - m^2)\lambda_0^a\right) \,. \tag{342}$$

♦ Note that $(c_0^a, \bar{c}_a^0)$ are free and integration produces the determinant $\det\,(\partial_t^2 - m^2)$. This exactly cancels its inverse coming from integration over λ_0^a.

Through these procedures, we finally obtain the familiar form

$$\boxed{\begin{aligned} S = \int d^4x \Big\{ & -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + B_a \left(\partial_\mu A_a^\mu + \frac{\alpha}{2} B_a \right) \\ & + i\bar{c}_a \partial_\mu (D^\mu c)^a \Big\} . \end{aligned} \qquad (343)}$$

where we have set $B_a \equiv \bar{\lambda}_a$.

5.2.6 A remark on non-relativistic gauges

When we carefully look at the proof of Ψ independence in the BFV formalism, we notice that it hinges on the **completeness of the ghost and the Lagrange multiplier system as dynamical variables.** This means that **we must keep $\dot{\lambda}^\alpha$ piece in Θ^α.**

However, **once Ψ independence is proven, we can go to the non-relativistic gauges such as the Coulomb gauge by a limiting procedure.**

Specifically, we take

$$\Theta^{\alpha} = \frac{1}{\xi}\dot{\lambda}^{\alpha} + \mathcal{F}^{\alpha}\,, \tag{344}$$

and consider the limit $\xi \longrightarrow \infty$. In this case, we have $(1/\xi)\bar{\lambda}_{\alpha}\dot{\lambda}^{\alpha}$, and this leads to the following changes:

$$\begin{aligned} \{\lambda^{\alpha}, \bar{\lambda}_{\beta}\} \;&=\; \xi\delta^{\alpha}_{\beta}\,, & (345)\\ i\wp_{\alpha}\tilde{\eta}^{\alpha} \;&\Longrightarrow\; i\xi\wp_{\alpha}\tilde{\eta}^{\alpha}\,, & (346)\\ \tilde{\eta}^{\alpha} \;&=\; \frac{i}{\xi}(D_0(\lambda)\eta)^{\alpha}\;. & (347) \end{aligned}$$

So in the limit $\xi \longrightarrow \infty$, the action becomes

$$S = \int dt \left\{ p_i \dot{q}^i - H_0 + \lambda^\alpha T_\alpha + \bar{\lambda}_\alpha \mathcal{F}^\alpha - i \bar{c}_\alpha \{\mathcal{F}^\alpha, T_\beta\} c^\beta \right\}. \tag{348}$$

The differences from the previous case:

- The time derivative terms for the ghosts vanished
- The gauge constraint does not contain $\dot{\lambda}^\alpha$ any more.

This scheme can readily be applied to the **Coulomb gauge** case, for example, where

$$\mathcal{F}^a = \partial_i A^i_a + \frac{\alpha}{2} \bar{\lambda}_a . \tag{349}$$

5.2.7 BRST transformation for the Lagrangian formulation

The BRST transformations of the fields were given **before** the integration over the ghosts $\wp_\alpha$ and $\tilde{\boldsymbol{\eta}}$.

After the integration, the space of fields becomes smaller and **we must rederive the BRST transformation property of the fields again**.

We write the Lagrangian density as

$$\begin{aligned} \boldsymbol{\mathcal{L}} \; = \; & -\frac{1}{4}\boldsymbol{F}^{\mu\nu}\cdot\boldsymbol{F}_{\mu\nu}+\boldsymbol{B}_a(\boldsymbol{\partial}_\mu\boldsymbol{A}_a^\mu+\frac{\boldsymbol{\alpha}}{2}\boldsymbol{B}_a) \\ & -i\boldsymbol{\partial}_\mu\bar{\boldsymbol{c}}_a\boldsymbol{D}^\mu\boldsymbol{c}^a \end{aligned} \tag{350}$$

where $\quad \boldsymbol{B}_a \equiv \bar{\boldsymbol{\lambda}}_a =$ Nakanishi-Lautrup field

□ **Conjugate Momenta**:

Let us apply the Dirac's method to this system. The part of the Lagrangian relevant for determining the conjugate momenta is

$$\mathcal{L}_0 = -\tfrac{1}{2}F^{0i} \cdot F_{0i} + B_a \dot{A}^0_a - i\partial_0 \bar{c}_a (D^0 c)^a \tag{351}$$

From this we read off

$$\begin{aligned}
\phi_1^a &\equiv \Pi_B^a = 0 \qquad \text{primary constraint} \\
\phi_2^a &\equiv \Pi_a^0 - B_a = 0 \qquad \text{primary constraint} \\
\Pi_a^i &= -F_a^{0i} = F_{0i}^a \\
\Pi_{c^a} &= -i\dot{\bar{c}}_a \qquad \text{using right derivative} \\
\Pi_{\bar{c}_a} &= +i(D^0 c)^a \qquad \text{using right derivative}
\end{aligned}$$

♦ Note that **for anti-commuting fields**, we must be careful about the differentiation.

The ordering leading to the standard definition of Poisson bracket is $p\dot{q}$. Thus we must differentiate with respect to $\dot{q}$ **from right**.

The two primary constraints ϕ_i^a are actually second class:

$$\{\phi_1^a(x), \phi_2^b(y)\} = \{\Pi_B^a(x), \Pi_b^0(y) - B_b(y)\} = -\delta_b^a(\vec{x} - \vec{y}) \neq 0$$

It is easy to check that **no secondary constraints arise.**

□ Quantization:

Because of these constraints, in A_0-B sector, we must compute the Dirac bracket. It is easy to convince oneself that the only non-vanishing bracket is

$$\begin{aligned}\left\{B_a(x), A_0^b(y)\right\}_D &= \left\{B_a(x), A_0^b(y)\right\} \\ &= -\int d^3z \int d^3w \left\{B_a(x), \phi_1^a\right\} \\ &\quad \times(-1)\delta_c^d\delta(\vec{z}-\vec{w})\left\{\phi_2^d, A_0^b(y)\right\} \\ &= -\delta_a^b\delta(\vec{x}-\vec{y})\end{aligned}$$

Therefore we find the following operator quantitzation rules:

$$\begin{aligned}\left[A_i^a(x), \Pi_b^j(y)\right]_{ET} &= i\delta_b^a\delta(\vec{x}-\vec{y}) \\ \left[A_0^a(x), B_b(y)\right]_{ET} &= i\delta_b^a\delta(\vec{x}-\vec{y}) \\ \{c^a(x), \Pi_{c^b}\}_{ET} &= i\delta_b^a\delta(\vec{x}-\vec{y}) \\ \{\bar{c}_a(x), \Pi_{\bar{c}_b}\}_{ET} &= i\delta_b^a\delta(\vec{x}-\vec{y})\end{aligned}$$

□ BRST Transformation:

Let us recall the BRST transformation rule for A_i^a in the Hamiltonian formalism. (To conform to the usual convention, we redefine it with a minus sign.)

$$\begin{aligned}
\delta_B A_i^a &\equiv -\left\{A_i^a(x), \Omega\right\} \\
&= -\int d^3y \left\{A_i^a(x), \mathcal{G}_b(y)\right\}_{ET} c^b(y) \\
&= -\int d^3y (\delta_{ab}\partial_i^y + g f_{abc} A_i^c(y))\delta(\vec{x} - vecy) c^b(y) \\
&= +\partial_i c^a(x) + g f_{abc} A_i^b(x) c^c(x) \\
&= (D_i c)^a = \partial_i c^a + g(A_i \times c)^a
\end{aligned}$$

The most important property of the BRST transformation is its nilpotency. So to find the transformation rule for c^a we impose

$$\begin{aligned}
0 &= \delta_B \delta_B A_i^a \\
&= \partial_i \delta_B c^a + g(\delta_B A_i \times c)^a + g(A_i \times \delta_B c)^a \\
&= \partial_i \delta_B c^a + g(\partial_i c \times c)^a + g(A_i \times \delta_B c)^a + g^2((A_i \times c) \times c)^a
\end{aligned}$$

This is satisfied if we define

$$\boldsymbol{\delta_B c^a \equiv -\tfrac{1}{2} g (c \times c)^a} \tag{352}$$

Then the RHS becomes

$$\boldsymbol{-\tfrac{1}{2} g \partial_i (c \times c) + g(\partial_i c \times c) + g^2 (A_i \times c) \times c - \tfrac{1}{2} g^2 (A_i \times (c \times c))}$$

This vanishes using the Jacobi identity.

♦ Since the above derivation did not depend on the spatial nature of the index $\boldsymbol{i}$ of $\boldsymbol{A_i^a}$, covariantization of the rule for the gauge potential works and we thus define

$$\boldsymbol{\delta_B A_\mu^a \equiv (D_\mu c)^a} \tag{353}$$

The nilpotency on $\boldsymbol{c}^a$ itself can be easily checked using the Jacobi identity:

$$\begin{aligned}
\boldsymbol{\delta_B \delta_B c} &= -\tfrac{1}{2}\boldsymbol{g\delta_B(c \times c)} \\
&= -\tfrac{1}{2}\boldsymbol{g(\delta_B c \times c - c \times \delta_B c)} \\
&= \frac{1}{4}\boldsymbol{g^2((c \times c) \times c - c \times (c \times c))} \\
&= \tfrac{1}{2}\boldsymbol{g^2(c \times c) \times c = 0} \quad \text{from Jacobi}
\end{aligned}$$

Now we come to the transformation on $\boldsymbol{\bar{c}_a}$ and $\boldsymbol{B_a}$. Since they formed a doublet in the Hamiltonian formalism, we take

$$\boldsymbol{\delta_B \bar{c}_a} = \boldsymbol{iB_a} \tag{354}$$

$$\boldsymbol{\delta_B B_a} = \boldsymbol{0} \tag{355}$$

Then $\delta_B^2 = 0$ **on all fields.**

□ Ghost-Gauge-Fixing Term:

We take the gauge fixing functin to be of the form

$$\mathcal{F}_a = \partial_\mu A_a^\mu + \frac{\alpha}{2} B_a \tag{356}$$

and write the BRST-invariant ghost-gauge-fixing Lagrangian as

$$\begin{aligned} \mathcal{L}_{Gh+GF} &= -i\delta_B(\bar{c}_a \mathcal{F}_a) \\ &= B_a(\partial_\mu A_a^\mu + \frac{\alpha}{2} B_a) + i\bar{c}_a \partial_\mu (D^\mu c)^a \end{aligned} \tag{357}$$

This is indeed the standard form. Note the i in front of the ghost term. Because of this, the **hermiticity of the ghosts** should be assigned as

$$c_a^\dagger = c_a\,, \qquad \bar{c}_a^\dagger = \bar{c}_a \tag{358}$$

□ Formalism without B_a Field:

In developing perturbation theory, it is often more convenient to integrate out the B_a field. Then we get

$$B_a(\partial_\mu A_a^\mu + \frac{\alpha}{2}B_a) \longrightarrow -\frac{1}{2\alpha}(\partial_\mu A_a^\mu)^2$$

As B_a transformed non-trivially under BRST, **we must modify the transformation law for $\bar{c}_a$** such that new $\mathcal{L}_{Gh+GF}$ should be BRST invariant. Thus require

$$\begin{aligned} 0 &= \delta_B\left(-\frac{1}{2\alpha}(\partial_\mu A_a^\mu)^2 + i\bar{c}_a(D^\mu c)^a\right) \\ &= -\frac{1}{\alpha}(\partial_\mu A_a^\mu)\partial_\mu(D^\mu c)^a + i\delta_B\bar{c}_a\partial_\mu\partial_\mu(D^\mu c)^a \end{aligned}$$

(Note $\delta_B(D^\mu c)^a = \delta_B^2 A_a^\mu = 0$.) From this we read off

$$\delta_B\bar{c}_a = -\frac{i}{\alpha}\partial_\mu A_a^\mu \tag{359}$$

which is the familiar rule.