Brane dynamics and Solvable systems[1]

Univ. of Tokyo, Yutaka Matsuo

1 Introduction

A critical difference between classical particles and D-branes comes from the fact that brane motion is described by the matrices instead of commuting variables. In this language, the actual location of i-th brane is described by the eigenvalues of embedding matrices A_μ [1][2] (μ runs over the indices of transverse dimensions.) The critical difference from classical particle equation appears when two branes collide, namely some of the eigenvalues coincide. In this case, one can not gauge away the off diagonal components of matrices and they are identified with the directions from which branes approach each other.

In the following, we restrict ourselves to 4 transverse dimensions and write the matrices along these directions in complex form,

$$B_1 = X_1 + iX_2 \quad , \; B_2 = X_3 + iX_4 \quad . \tag{1}$$

Equation of motion in terms of these new coordinates becomes, [3]

$$[B_1, B_2] = [B_1^\dagger, B_2^\dagger] = 0 \qquad \sum_{\alpha=1,2} [B_\alpha, B_\alpha^\dagger] = 0 \quad . \tag{2}$$

When N 0-branes approaches each other at $z_1 = z_2 = 0$ Jordan standard form of these matrices become,

$$B_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & & 0 \end{pmatrix} \qquad B_2 = \sum_{i=1}^{N-1} a_i B_1 \tag{3}$$

For the particle moduli space, this colliding limit is the singularity of the moduli space. On the other hand, as we saw, in the brane moduli space,

[1] Talk presented at "Komaba Workshop 99" (January 5-6, 1999)

the singularity is blown up and has moduli with $N-1$ complex dimension. Namely, at the singularity, $N-1$ cycle is naturally attached.

Armed with this knowledge, it is rather easy to get homology group of moduli space. For $N=1$ case, the moduli space is trivial (C^2) and homology group is nontrivial only for $i=0$. For $N=2$ case, on the other hand, we get nontrivial 2-cycle from coinciding two branes. More generally, one may give the generating function of Poincare polynomial in the following interesting form (so called Götche's formula)[3],

$$\begin{aligned}\sum_{n=0}^{\infty} q^n P_t((C^2)^{[n]}) &= \prod_{n=1}^{\infty} \frac{1}{1-t^{2n-2}q^n} \qquad (4)\\ &= 1+q+(t^2+1)q^2+(t^4+t^2+1)q^3+\cdots. \qquad (5)\end{aligned}$$

Very interestingly, the right hand side of this formula is the partition function of a free boson.

This free boson was introduced by Nakajima in mathematical literature. The creation operator a_{-n} may be interpreted as attaching collapsed n 0-branes at one point homo-logically. In physical terminology, it is nothing but the second quantized operators for (topological) p-branes.

Let us proceed to explicit description of the homology class. The method we take here is to employ Lifshitz fixed point theorem. Consider the T^2 flow in the moduli space,

$$(B_1, B_2) \to (z_1 B_1, z_2 B_2). \qquad (6)$$

Fixed points under this flow is given by following equations,

$$z_1 B_1 = g B_1 g^{-1} \qquad z_2 B_2 = g B_2 g^{-1} \qquad (7)$$

for some fixed $g \in G$. Its infinitesimal form may be written as,

$$E_1 B_1 = [\phi, B_1] \qquad E_2 B_2 = [\phi, B_2] \qquad (8)$$

(by writing $z_i = e^{E_i t}$ and $g = e^{\phi t}$).

For $(C^2)^{[N]}$, fixed points are labeled by Young diagram with N boxes. (Let us denote it as p_Y for Young diagram Y.) The tangent space at this fixed point p_Y is spanned by basis $\vec{v}_i$ where i runs over each boxes in Y. Let $\rho_\alpha(i)$ be the index associated with the box in right (down) neighbor for $i=1$ ($i=2$). The action of B_α ($\alpha=1,2$) is defined in this basis as $B_\alpha \vec{v}_i = \vec{v}_{\rho_\alpha(i)}$. ($B_\alpha \vec{v}_i = 0$

if $\rho_\alpha(i)$ is absent). With ϕ defined by $\phi = diag(\phi_1, \cdots, \phi_N)$, $\phi_{\rho_\alpha(i)} - \phi_i = E_\alpha$, B_α and ϕ satisfy the fixed point equation. The eigenvalue for the infinitesimal action may be written in the following form,

$$\frac{\partial}{\partial t}\vec{v}_i = h_i \vec{v}_i \tag{9}$$

with h_i takes its value either $h_*(i) = E_1(\ell(i)+1) - E_2 a(i)$, or $h^*(i) = -E_1\ell(i) + E_2(a(i)+1)$. By counting the number of positive eigenvalues, we can reproduce Götsche's formula.

At this point, we summarize a few questions:

- We encountered free boson to describe homology class. What is the role of fermion (or more generally vertex operators) then?
- With free boson, we get a natural definition of Virasoro generators. What is their physical (or mathematical) rôle?
- Many solvable systems have the structure of the free boson as the origin of solvability. (Examples: KP hierarchy, Calogero-Sutherland system) One may expect a relation between brane moduli space and solution space of these equations (typically Grassmannian manifold).

2 Matrix integration as Cohomological field theory

Recently Moore, Nekrasov and Shatashivilli (MNS) [4] remarked that the type IIB matrix integration,

$$\begin{aligned} I_D(N) &\propto \int d^D X d^{\tilde{D}}\Psi e^{-S} \\ S &= \frac{1}{g}\Big(\frac{1}{4}\sum_{\mu,\nu=1\cdots D} Tr[X_\mu, X_\nu]^2 + \frac{i}{2}\sum_{\mu=1}^{D} Tr(\bar{\Psi}\Gamma^\mu[X_\mu,\Psi])\Big). \end{aligned} \tag{10}$$

Here we generalize type IIB matrix model to $D = 4, 6, 10$. Corresponding degree of freedom for fermion is $\tilde{D} = 4, 8, 16$.

To get correspondence with cohomological field theory, we rewrite integration in "light cone gauge" variables,

$$\begin{aligned}
\phi &= X_{D-1} + iX_D \\
B_j &= X_{2j-1} + iX_{2j} \quad (j = 1, \cdots, \frac{D}{2} - 1) \\
\Psi &\to (\psi_j, \psi_j^\dagger), \vec{\chi}, \eta
\end{aligned} \tag{11}$$

$vec\chi$ describes degree of freedom of "equation of motion" and has respectively 1,3,7 dimensions for $D = 4, 6, 10$.

The equation of motion for transverse degree of freedom becomes,

$$\begin{aligned}
D = 4 \quad \vec{\mathcal{E}} &= [B_1, B_1^\dagger] = 0 \\
D = 6 \quad \vec{\mathcal{E}} &= (\sum_\alpha [B_\alpha, B_\alpha^\dagger], [B_1, B_2], [B_1^\dagger, B_2^\dagger]) \\
D = 10 \quad \vec{\mathcal{E}} &= (\sum_\alpha [B_\alpha, B_\alpha^\dagger], [B_i, B_j] + \frac{1}{2}\epsilon_{ijkl}[B_k^\dagger, B_l^\dagger])
\end{aligned} \tag{12}$$

The moduli space for each case has respectively complex ($D = 4$), quoternion ($D = 6$), and octonion ($D = 10$) structure.

One of the critical observation in MNS was that the action has a fermionic symmetry:

$$\begin{aligned}
QB_i = \psi_i, &\qquad Q\psi_i = [\phi, B_i] \\
Q\vec{\chi}_i = \vec{H}, &\qquad Q\vec{H} = [\phi, \vec{\chi}] \\
Q\bar{\phi} = \eta &\qquad Q\eta = [\phi, \bar{\phi}] \\
Q\phi = 0 &
\end{aligned} \tag{13}$$

The action itself is then written as Q-exact form as $S = Q(Tr\frac{1}{16g}\eta[\phi, \bar{\phi}] - \cdots)$. Partition function is characterized by its value at the saddle points. At this point, however, the saddle point equation (for example $[\phi, B_i] = 0$) has flat directions and naive evaluation of partition function diverges. To regularize the integration, MNS introduced a deformation of fermionic symmetry by using deformation matrices, $\epsilon = J(E)$ for $D = 4$, $= diag(J(E_1), J(E_2))$ for $D = 6$ and $= diag(J(E_1 + E_2), J(E_2 + E_3), J(E_3 + E_1))$ for $D = 10$ with $J(E) = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. By introduction of these parameters, equations of motion for B_i and $\vec{H}$ are modified to,

$$[\phi, B_i] + B_i T_v(\epsilon) = 0 \quad [\phi, \vec{\chi}] + T_s(\epsilon)\vec{\chi} = 0. \tag{14}$$

T_v (resp.T_s) are vector (1,3,7 dimensional) representation of $O(2), O(4), O(6)$. We may note that these equations are the fixed point equations for T^1, T^2, T^3 flows which we encountered previously.

MNS claimed that one may evaluate explicitly the integration over η, $\bar{\phi}$, $\vec{\chi}$, $\vec{H}$, B_i, ψ_i at these fixed points while leaving only one matrix integration with respect to ϕ,

$$\begin{aligned} I(N) &= (C(E))^{N-1}\frac{1}{(N-1)!}\int D\phi \prod_{i\neq j}\frac{P(\phi_{ij})}{Q(\phi_{ij})} && (15)\\ C(E) &= \frac{(E_1+E_2)(E_2+E_3)(E_3+E_1)}{E_1E_2E_3E_4}, (D=10) \\ & \quad (E_4 = -E_1 - E_2 - E_3) \\ &= \frac{E_1+E_2}{E_1E_2} \quad (D=6) \\ &= 1/E \quad (D=4) && (16)\\ P(\phi) &= \phi(\phi+E_1+E_2)(\phi+E_2+E_3)(\phi+E_1+E_3) \quad (D=10) \\ &= \phi(\phi+E_1+E_2) \quad (D=6) \\ &= \phi && (17)\\ Q(\phi) &= \prod_{\alpha=1}^{4}(\phi+E_\alpha+0) \quad (D=10) \\ &= \prod_{\alpha=1}^{2}(\phi+E_\alpha+0) \quad (D=6) \\ &= \phi+E && (18) \end{aligned}$$

Here the matrix integration with respect to off-diagonal components of ϕ is already performed and integration by diagonal components are left. With some consideration, it is not difficult to notice that these integrations can be evaluated at simple poles at the fixed points of $U(1)$ actions if ϕ is identified with infinitesimal gauge transformation. More explicitly, we write down generalized "Young diagram" with $d = 1, 2, 4$ dimensions with N-boxes. We put label $i = 1 \cdots N$ into each box in the diagram and write $\rho_\alpha(i)$ as the label of the neighboring box to the α's direction. Then the ϕ at the simple pole satisfies, $[\phi, B_\alpha] + E_\alpha B_\alpha = 0$ where $\phi_{\rho_\alpha(i)} = \phi_i + E_\alpha$ and $(B_\alpha)_{ij} = \delta_{j\rho_\alpha(i)}$. $D = 6$ case is exactly the situation which we encountered in the introduction.

The remaining task is to evaluate the residue explicitly. For $D = 4$ case, there is only one diagram and its residue is simply $1/N^2$. For $D = 6$ case, the residue for Young diagram Y is,

$$R_Y = (-1)^{N-1} E_1 E_2 \frac{\prod_{(\alpha,\beta)\neq(1,1)}((\alpha-1)E_1 + (\beta-1)E_2)(\alpha E_1 + \beta E_2)}{\prod_{(\alpha,\beta)} h_*(\alpha,\beta) h^*(\alpha,\beta)} \tag{19}$$

It has rather complicated form but the summation over Young diagram with small N can be carried out explicitly and lead us to conjecture, $\sum_{|Y|=N} R_Y = 1/N$.

For $D = 10$ case, we have no explicit formula for the residue of each diagram. Some explicit evaluation for lower N are again possible. Judging from these explicit values, one may conjecture,

$$\sum_{|Y|=N} R_Y = \sum_{d|N} \frac{1}{d^2} \tag{20}$$

3 Correspondence with solvable system

As we pointed out in the introduction, it is quite natural to pursue the structure of exactly solvable models in matrix integration. In this viewpoint, the cohomological field theory of MNS gives an ideal starting point. One of the most natural ideas is to identify $\sum_{i=1}^N \phi_i^n$ as Miwa variable of KP hierarchy. This is the step taken in the "old" matrix model and pursued in this context by Kazakov,Kostov and Nekrasov. In MNS matrix integration, we have a factor of $\Delta \equiv \prod_{i\neq j}(\phi_i - \phi_j)$. It corresponds to the free fermionic as $\psi(\phi_1)\cdots\psi(\phi_N)$. In this language, free boson oscillator is Miwa variable itself. Starting from MNS integration, one may obtain the generating function for correlation function explicitly.

However, we have to comment that this free fields are not the Nakajima's free fields and does not describe the cohomological structure of the moduli of brane motion. First of all, KKN oscillators are the moment of gauge transformation which is not physical. Second, it does not change the size of matrix and does not create degenerate branes. On the other hand, Nakajima's oscillator, does describe the creation/annihilation of branes and does change the size of matrices.

In the following we would like to argue that Nakajima's oscillator when written in the form of fermionic field should be identified with the inter-winner between fixed points of the moduli space [5]. In this language, we would like to interpret the residue factor R_Y as inner product between fermionic states created from vacuum.

The first reasoning is comparison of moduli space. In the solvable systems as KP hierarchy or Calogero-Sutherland system, the moduli of solutions are given by certain Grassmannian with infinite dimensions. Instead of finite dimensional torus action, we have infinite dimensional action generated by zero modes of $W_{1+\infty}$ algebra. The fixed points of T^∞ action are described by states with finite free fermion excitation and can be labeled by Young diagram. The dictionary between Young diagram and Fock space is described by Maya diagram.

Therefore, these two moduli spaces, (Grassmannian and Hilbert scheme) have some properties in common although they are quite different spaces. To pursue the connection further one may note that the tangent space at the fixed point have similar structure. We remarked before that T^2 action at the fixed point have eigenvalues $h_*(i)$ and $h^*(i)$. Let us compare it with eigenvalues of L_0 (Virasoro zero mode) at the fixed point of Grassmannian. By the construction of fixed point as fermionic excited states, there is one to one correspondence between each box in the Young diagram and level lowering generator in $gl(\infty)$. The L_0 eigenvalues of these generators have the same form as h_* and h^* if we fix parameters as $E_2 = -E_1$. In this case, these parameters reduce to the hook length.

The second evidence comes from MNS's residue formula. If we set $\beta = -1$, only diagrams with one hook are non-vanishing and the residue reduce to $R_Y = 1/(\text{hook length} = N)^2$ which is independent of the shape. Let us assign each fixed point a fermionic eigenstate as described by Maya diagram and try to reproduce residue from inner product. We seek a level N state $|N>$ which satisfies, $<Y|N>= 0$ if Y has more than one hook and $<Y|N>=$ constant if Y has one hook. Such a state exist uniquely and is given by $|N>\propto a_{-N}|0>$. In this language, MNS formula should read, $R_Y = |<0|\frac{a_N}{N}|Y>|^2$. If we sum over N with weight q, we get

$$\begin{aligned} Z(q) &\equiv \sum_{N=0}^{\infty} (NZ_N) q^N = \sum_Y <0|\phi(q)|Y><Y|\phi(1)|0> \\ &= <0|\phi(q)\phi(1)|0> \end{aligned} \tag{21}$$

Weight factor transforms into the world sheet coordinates. Similarly Virasoro action is also given as reparametrization of the formal variable q,

$$q^{n+1}\frac{\partial}{\partial q}Z(q) =< 0|[L_n,\phi(q)],\phi(1)|0> . \tag{22}$$

Up to this point, we have to restrict ourselves to the special case $E_2 = -E_1$. For general $\beta = -E_2/E_1$, we remark that the weight factor h_* and h^* of T^2 action is identical with the weight factor of Jack polynomial (the eigenstates of Calogero-Sutherland system),

$$< J_Y, J_Y >= \prod_i h^*(i)h_*(i). \tag{23}$$

Parameter β is identical with the coupling constant of Calogero-Sutherland system. It is known that Jack polynomial J_Y can be written in terms of inter-wining operators, $e^{-\frac{1}{\sqrt{\beta}}\phi}$ and $e^{\sqrt{\beta}\phi}$ and can be interpretable as the singular vectors of W algebras. After some calculation, one may claim that the MNS residue formula is given in terms of Jack polynomials as,

$$\begin{aligned}
R_Y &= < 0|\frac{a_N}{N}|J_Y> \frac{1}{< J_Y|J_Y>} < J_Y|X_{-N}|0> \\
\sum_{n=0}^{\infty} q^n X_{-n}|0> &= \frac{1}{\gamma}e^{\gamma\phi(q)}|0> \\
\gamma &= -\sqrt{\beta}+1/\sqrt{\beta}
\end{aligned} \tag{24}$$

In this formula, it is clear that the independence of partition function of matrix model on β become the consequence of completeness of the basis spanned by Jack polynomials.

References

[1] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, Phys.Rev. D55 (1997) 5112-5128, hep-th/9610043.

[2] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, Nucl.Phys. B498 (1997) 467-491, hep-th/9612115.

[3] H. Nakajima, “Lectures on Hilbert schemes of points on surfaces”, (preprint, Oct. 1996) and references therein.

[4] G. Moore, N. Nekrasov, S. Shatashvili, “D-particle bound states and generalized instantons”, hep-th/9803265.

[5] Y. Matsuo, Mod.Phys.Lett.A13:2731-2742,1998, hep-th/9807085