Background Independence in Doubled Field Theory

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Works in collaboration with Chris Hull
Upcoming paper with C. Hull and Olaf Hohm.

Closed Strings on Toroidal backgrounds:

Questions:

* Field theory on doubled torus?
  Momentum + winding.. T-duality?

* Special geometry? Courant brackets?

* In string theory $g_{ij} + b_{ij} = \mathcal{E}_{ij}$
  is the natural variable.
  How does one write actions?

The full closed string field theory is a double field theory (T. Kugo + B.Z) but has two shortcomings:

(1) With infinite # of fields and nonpolynomial interactions, it is complicated

(2) It does not (yet!) have manifest background independence.

Thus look for a simpler subset of the theory to learn the lessons of double field theory and background independence.

Closed strings on a torus T^d

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij} \right)$$

$E_{ij} \equiv G_{ij} + B_{ij}$ both constant backgrounds

$X^i \equiv X^i + 2\pi$

$$\left(G_{ij} = \frac{R_i^2}{\alpha'}\delta_{ij} \right)$$

$$X^i(\tau,\sigma) = x^i + w^i\sigma + \tau G^{ij}(p_j - B_{jk}w^k) + \cdots$$

$$[x^i, p_j] = [\tilde{x}_j, w^i] = i\delta^i_j$$

$$p_i = \frac{1}{i}\frac{\partial}{\partial x^i} = \frac{1}{i}\partial_i$$

$$w^i = \frac{1}{i}\frac{\partial}{\partial \tilde{x}_i} = \frac{1}{i}\tilde{\partial}^i$$

w^i eigenvalues $m^i \in \mathbb{Z}$

p_i eigenvalues $n_i \in \mathbb{Z}$

States of the theory:

oscillators

$$\sum_{m^i, n_j} \xi_{i_1 \cdots i_p, j_1 \cdots j_\ell}(m^i, n_j)\,(\alpha^{i_1} \cdots \alpha^{i_p})\,(\bar{\alpha}^{j_1} \cdots \bar{\alpha}^{j_\ell})\,(c \cdots c)(b \cdots b)\,|m^i, n_j\rangle$$

winding + momentum

Natural fields are doubled:

$$\xi_{i_1 \cdots i_p, j_1 \cdots j_\ell}(\underset{\uparrow}{x}, \underset{\uparrow}{\tilde{x}}) \equiv \sum_{m^i, n_j} \xi_{i_1 \cdots i_p, j_1 \cdots j_p}(m, n)\, e^{i(m^i \tilde{x}_i + n_j x^j)}$$

gauge parameters doubled too!

In string theory fix the background E_{ij} and consider fluctuations

$$\mathcal{E}_{ij} = E_{ij} + e_{ij}(x, \tilde{x}) \quad \text{(Tentative)}$$

$$g_{ij} + b_{ij} = G_{ij} + B_{ij} + h_{ij}(x, \tilde{x}) + b_{ij}(x, \tilde{x})$$

Also dilaton field $d(x, \tilde{x})$

$$\sum_{p,w} e_{ij}(p,w)\, \alpha^i_{-1} \bar{\alpha}^j_{-1}\, c_1 \bar{c}_1\, |p,w\rangle$$

$$+ \, d(p,w)\left(c_{-1} c_{+1} - \bar{c}_{-1} \bar{c}_1\right) |p,w\rangle$$

Focus on this "massless sector" only but with fully doubled fields

$$N = \bar{N} = 1$$

All string states $|\psi\rangle$.

$$(L_0 - \bar{L}_0)|\psi\rangle = 0$$ Off shell constraint

$$\longrightarrow L_0 - \bar{L}_0 = N - \bar{N} - p_i w^i = 0$$

For our states $p_i w^i = 0$,

or $\quad p \cdot w = 0$

or $\quad \dfrac{\partial}{\partial x^i} \dfrac{\partial}{\partial \tilde{x}_i} \left[\text{fields}(x, \tilde{x})\right] = 0.$

Unavoidable constraint

$$\partial_k \tilde{\partial}^k \left\{ e_{ij}(x, \tilde{x}),\ d(x, \tilde{x}) \right\} = 0$$

The restriction to "massless" doubled fields is a truncation.

It is not a low-energy limit.

Open question if a full gauge invariant field theory exists with those degrees of freedom.

Attempt construction using the information from closed string field theory.

Cubic theory: around E_{ij}

$$\boxed{\begin{aligned} S = \int dx\,d\tilde{x}\,\Big[& \tfrac{1}{4} e_{ij} \Box e^{ij} + \tfrac{1}{4} (\bar{D}^j e_{ij})^2 + \tfrac{1}{4} (D^i e_{ij})^2 \\ & - 2d\, D^i \bar{D}^j e_{ij} - 4 d \Box d \\ & + \tfrac{1}{4} e_{ij} \Big((D^i e_{kl})(\bar{D}^j e^{kl}) - D^i e_{kl}\, \bar{D}^l e^{kj} \\ & \qquad - D^k e^{il}\, \bar{D}^j e_{kl} \Big) \\ & + \tfrac{1}{2} d \Big((D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \tfrac{1}{2} (D_k e_{ij})^2 \\ & \qquad + \tfrac{1}{2} (\bar{D}_k e_{ij})^2 + 2 e^{ij} \big(D_i D^k e_{kj} + \bar{D}_j \bar{D}^k e_{ik} \big) \Big) \\ & + 4 e_{ij}\, d\, D^i \bar{D}^j d + 4 d^2 \Box d + \mathcal{O}(f^4) \Big] \end{aligned}}$$

G^{ij} raises indices $\qquad e^{ik} = G^{ip} G^{kl} e_{pl}$

$$\Box = D^i D_i = G^{ij} D_i D_j$$

$$D_i \equiv \partial_i - E_{ik} \tilde{\partial}^k$$

$$\bar{D}_i \equiv \partial_i + E_{ki} \tilde{\partial}^k$$

Gauge transformations

$$\delta_\lambda e_{ij} = \bar{D}_j \lambda_i + \frac{1}{2}\left[(D_i \lambda^k) e_{kj} - (D^k \lambda_i) e_{kj} + \lambda_k D^k e_{ij}\right]$$

$$\delta_\lambda d = -\frac{1}{4} D.\lambda + \frac{1}{2}(\lambda . D) d$$

$$\delta_{\bar\lambda} e_{ij} = D_i \bar\lambda_j + \frac{1}{2}\left[(\bar D_j \bar\lambda^k) e_{ik} - (\bar D^k \bar\lambda_j) e_{ik} + \bar\lambda_k \bar D^k e_{ij}\right]$$

$$\delta_{\bar\lambda} d = -\frac{1}{4} \bar D . \bar\lambda + \frac{1}{2}(\bar\lambda . \bar D) d$$

In the products here, one must project to the kernel of $\partial \cdot \tilde\partial$

Theory also has $O(D,D)$ symmetry

$$h = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \in O(D,D) \text{ if}$$

$$h^t \eta h = \eta \quad \text{with} \quad \eta = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$$

$$E \longrightarrow E' = h(E) = (aE+b)(cE+d)^{-1}$$

$$X^M = \begin{pmatrix} \tilde x_i \\ x^i \end{pmatrix} \longrightarrow X'^M = \begin{pmatrix} \tilde x'_i \\ x'^i \end{pmatrix} = h \begin{pmatrix} \tilde x_i \\ x^i \end{pmatrix}$$

$$\partial^M = \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix} \qquad \eta^{MN} = \eta_{MN} = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{array}\right)$$

$\uparrow$ $O(D,D)$ invariant metric

$$\partial_M = \eta_{MN}\,\partial^N = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}$$

Our constraint is $\partial_M \partial^M A = 0$

$O(D,D)$ covariant:

$$\partial_i \tilde{\partial}^i A = 0 \quad \longrightarrow \quad \partial_i' \tilde{\partial}'^i A = 0$$

$O(D,D)$ transformations

$$X' = gX$$

$$e_{ij}(X) = M_i{}^k \bar{M}_j{}^l e'_{kl}(X')$$

$$d(X) = d'(X')$$

$$M = d^t - Ec^t$$

$$\bar{M} = d^t + E^t c^t$$

$$S(E, e_{ij}, d) = S(E', e', d')$$

Options

* Continue the construction to higher order
* Extract the lessons so far.

Will do the second!

Will impose a strong $O(D,D)$ invariant constraint (in addition to $\partial_M \partial^M A = 0$) and will finish the higher order construction [Related to work of Siegel]

For A, B, such that $\partial^M \partial_M A = \partial^M \partial_M B = 0$

$$\partial^M \partial_M (AB) = 2\, \partial^M A\, \partial_M B \neq 0, \text{ in general}$$

Assume (**) $\boxed{\partial^M A\, \partial_M B = 0 \quad \text{for any fields } A, B}$

thus all fields and all products in the kernel of $\partial \cdot \tilde{\partial}$.

Claim: for fields that satisfy (**) there is a duality frame $(\tilde{x}'_i, x'^i)$ in which fields do not depend on $\tilde{x}'_i$ $(\tilde{\partial}'^i \equiv 0)$

This duality frame need not be specified explicitly; the constraint can be imposed without doing so.

$$\boxed{\text{The resulting action will have } O(D,D) \text{ symmetry, will involve } x, \tilde{x}, \text{ but it is a } O(D,D) \text{ covariantization of a theory that in some dual frame is not doubled.}}$$

The gauge algebra will be novel; Courant brackets (Hull + B.Z)

The action can be made background independent and duality invariant (Hohm, Hull, B-Z to appear)

The gauge transformations can now be completed to "all orders"

$$\delta_\lambda e_{ij} = D_i \bar\lambda_j + \bar D_j \lambda_i$$
$$+ \frac{1}{2}(\lambda\cdot D + \bar\lambda\cdot\bar D)\, e_{ij}$$
$$+ \frac{1}{2}(D_i\lambda^k - D^k\lambda_i)\, e_{kj} - e_{ik}\frac{1}{2}(\bar D^k\bar\lambda_k - \bar D_j\bar\lambda^k)$$
$$- \frac{1}{4}\, e_{ik}\,(D^l\bar\lambda^k + \bar D^k\lambda^l)\, e_{lj}$$

Better variables:

$$\xi^i = \frac{1}{2}(\lambda^i + \bar\lambda^i) \qquad \tilde\xi_i = \frac{1}{2}(-E_{ji}\lambda^j + E_{ij}\bar\lambda^j)$$

$O(D,D)$ doublet
$$\Sigma^M = \begin{pmatrix}\tilde\xi_i \\ \xi^i\end{pmatrix}$$

Trivial gauge parameters

$$\Sigma^M = \begin{pmatrix}\partial_i\chi \\ \tilde\partial^i\chi\end{pmatrix} = \partial^M\chi$$

Algebra of gauge transformations

$$[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1,\Sigma_2]_C}$$

$$\boxed{([\Sigma_1,\Sigma_2]_C)^M = \Sigma^N_{[1}\partial_N\Sigma^M_{2]} - \frac{1}{2}\eta^{MN}\eta_{PQ}\,\Sigma^P_{[1}\partial_N\Sigma^Q_{2]}}$$

(Also Siegel)

What does this give explicitly

write $\Sigma \equiv \xi + \tilde{\xi}$ $\quad \xi^i(x,\tilde{x})\,,\ \tilde{\xi}_i(x,\tilde{x})$

$$\boxed{\begin{aligned}[\xi_1+\tilde{\xi}_1,\ \xi_2+\tilde{\xi}_2] = {} & [\xi_1,\xi_2] + \mathcal{L}_{\tilde{\xi}_1}\xi_2 - \mathcal{L}_{\tilde{\xi}_2}\xi_1 \\ & -\tfrac{1}{2}\tilde{d}\left(\tilde{\iota}_{\tilde{\xi}_1}\xi_2 + \tilde{\iota}_{\tilde{\xi}_2}\xi_1\right) \\ & + [\tilde{\xi}_1,\tilde{\xi}_2] + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 \\ & -\tfrac{1}{2}d\left(\iota_{\xi_1}\tilde{\xi}_2 - \iota_{\xi_2}\tilde{\xi}_1\right)\end{aligned}}$$

As usual $\quad [\xi_1,\xi_2]^i = \xi_{[1}^k\partial_k\xi_{2]}^i$

but now $\quad [\tilde{\xi}_1,\tilde{\xi}_2]_i \equiv \tilde{\xi}_{[1k}\tilde{\partial}^k\xi_{2]i}$

$$\tilde{d}f \equiv (\tilde{\partial}^k f)\,d\tilde{x}_k$$

Above is a Courant bi-algebroid

Let $x^i, \tilde{x}_i$ denote the frame where there is no winding $\tilde{\partial}^i \equiv 0$. Then

$$\boxed{\begin{aligned}[\xi_1+\tilde{\xi}_1,\ \xi_2+\tilde{\xi}_2] = {} & [\xi_1,\xi_2] \\ & + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 \\ & -\tfrac{1}{2}d\left(\iota_{\xi_1}\tilde{\xi}_2 - \iota_{\xi_2}\tilde{\xi}_1\right)\end{aligned}}$$

This is the well-known Courant bracket of special geometry.

For a theory of a metric g and a KR field b

gauge transform.
$$\begin{cases} \delta_{\xi+\tilde{\xi}}\, g = \mathcal{L}_\xi\, g \\ \delta_{\xi+\tilde{\xi}}\, b = \mathcal{L}_\xi\, b + d\tilde{\xi} \end{cases}$$

The gauge algebra is

$$\begin{aligned}
[\xi_1+\tilde{\xi}_1,\ \xi_2+\tilde{\xi}_2] = {} & [\xi_1, \xi_2] \\
& + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 \\
& - \tfrac{1}{2}\beta\, d\left(\iota_{\xi_1}\tilde{\xi}_2 - \iota_{\xi_2}\tilde{\xi}_1\right)
\end{aligned}$$

β cannot be fixed. (exact 1-form, trivial gauge sym)

T-duality covariance fixes it to $\beta = 1$! This is the "Courant bracket"

(a bracket with an extra automorphism)

Jacobi identity violated by an exact one-form!

How about the action?

We have an action $S(E_{ij}\,;\,e_{ij})$

We did a background independence analysis: Let χ_{ij} be a constant

$$S\left(E_{ij}-\chi_{ij}\,;\,e_{ij}+\chi_{ij}-\tfrac{1}{2}\left(\chi_i{}^k e_{kj}+\chi^k{}_j e_{ik}\right)\right)$$

$$= S\left(E_{ij}\,;\,e_{ij}\right)$$

$$\delta_\chi E_{ij} = -\chi_{ij}$$

$$\delta_\chi e_{ij} = \chi_{ij}-\tfrac{1}{2}\left(\chi_i{}^k e_{kj}+\chi^k{}_j e_{ik}\right)$$

Can then verify that

$$\boxed{\mathcal{E}_{ij} = E_{ij}+e_{ij}+\tfrac{1}{2}e_i{}^k e_{kj}+\mathcal{O}(e^3)}$$

is background independent!

$$\delta_\chi \mathcal{E}_{ij} = 0 \quad \left(\text{up to } \mathcal{O}(e^3)\right)$$

To all orders

$$\mathcal{E}_{ij} = E_{ij}+\left(1-\tfrac{1}{2}e\right)^{-1}{}_i{}^k e_{kj}$$

The action $S(\mathcal{E}, d)$ in terms of $\mathcal{E}_{ij}$

and
$$\mathcal{D}_i = \partial_i - \mathcal{E}_{ik}\tilde{\partial}^k$$
$$\bar{\mathcal{D}}_i = \partial_i + \mathcal{E}_{ki}\tilde{\partial}^k$$

that reproduces to $\mathcal{O}(e^3)$ the old action $S(E; e, d)$ is:

$$\boxed{\begin{aligned} S = \int dx\, d\tilde{x}\; e^{-2d}\Big[& -\tfrac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl}\, \mathcal{D}_p \mathcal{E}_{ij} \\ & + \tfrac{1}{4} g^{kl}\left(\mathcal{D}^j \mathcal{E}_{ik}\, \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki}\, \bar{\mathcal{D}}^i \mathcal{E}_{lj} \right) \\ & + \mathcal{D}^i d\, \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\, \mathcal{D}^j \mathcal{E}_{ji} \\ & + 4 \mathcal{D}^i d\, \mathcal{D}_i d \Big] \end{aligned}}$$

But can't add terms with 2-derivatives without spoiling this $S(\mathcal{E}, d)$

Claim this is the <u>exact action</u>!!

Gauge invariance:

$$\boxed{\begin{aligned} \delta \mathcal{E}_{ij} = & \ \mathcal{D}_i \tilde{\xi}_j - \mathcal{D}_j \tilde{\xi}_i + \xi^M \partial_M \mathcal{E}_{ij} \\ & + \mathcal{D}_i \xi^k \mathcal{E}_{kj} + \bar{\mathcal{D}}_j \xi^k \mathcal{E}_{ik} \end{aligned}}$$

$O(D,D)$ invariance:

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad X' = hX$$

$$\mathcal{E}'(X') = \left(a\,\mathcal{E}(X) + b\right)\left(c\,\mathcal{E}(X) + d\right)^{-1}$$

$$d'(X') = d(X)$$

$$S(\mathcal{E}', d') = S(\mathcal{E}, d)$$

$\tilde{\partial}$ - Expansion:

$$\delta = \delta^{(0)} + \delta^{(1)}$$

$$\boxed{\begin{aligned}
& \delta \mathcal{E}_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i \\
\mathcal{L}_{\xi}\mathcal{E} \quad \delta^{(0)} \quad & + \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \mathcal{E}_{kj} + \partial_j \xi^k \mathcal{E}_{ik} \\
\tilde{\mathcal{L}}_{\tilde{\xi}}\mathcal{E} \quad & + \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}_i \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \mathcal{E}_{ik} \\
\delta^{(1)} \quad & + \mathcal{E}_{ik}\left(\tilde{\partial}^q \xi^k - \tilde{\partial}^k \xi^q\right) \mathcal{E}_{qj}
\end{aligned}}$$

For the action:

$$S = S^{(0)} + S^{(1)} + S^{(2)}$$

$$S^{(0)} = \int dx\, d\tilde{x}\, e^{-2d} \Big[-\tfrac{1}{4} g^{ik} g^{jl} g^{pq} \big(\partial_p \mathcal{E}_{kl} \partial_q \mathcal{E}_{ij} - \partial_i \mathcal{E}_{lp} \partial_j \mathcal{E}_{kq} - \partial_i \mathcal{E}_{pl} \partial_j \mathcal{E}_{qk} \big) + 2 \partial^i d\, \partial^j g_{ij} + 4 \partial^i d\, \partial_i d \Big]$$

After field redefinition and total derivatives: $e^{-2\phi}\sqrt{-g} = e^{-2d}$

$$S^{(0)} = \int dx\, d\tilde{x}\, e^{-2\phi} \sqrt{g} \left(R + 4 (\partial\phi)^2 - \tfrac{1}{12} H^2 \right) \quad \checkmark$$

$$= S(\mathcal{E}, d, \partial)$$

Then find that

$$S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde{\partial}) \;!! \qquad h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{E}' = \mathcal{E}^{-1}, \quad x' = \tilde{x}, \quad \tilde{x}' = x$$

$S^{(1)}$ is an "intertwiner"

$$S^{(1)} = \int dx\, d\tilde{x}\, e^{-2d} \Big[\cdots b\, \tilde{\partial} b\, \partial b + \cdots b\, \partial d\, \tilde{\partial} d + \cdots \Big]$$

Is there a curvature scalar: R $\quad \delta_\Sigma R = \Sigma^M \partial_M R$

$$R(e,d) = 4D^2 d + D^i \bar{D}^j e_{ij} + \cdots$$

Is there a background independent $R(\mathcal{E},d)$ version?

$$\boxed{\begin{aligned} S &= \int e^{-2d} \mathcal{R}(\mathcal{E},d) \\ &= \int e^{-2d} \Big[2(\nabla^i \mathcal{D}_i d + \bar{\nabla}^i \mathcal{D}_i d) \\ &\quad + \tfrac{1}{2}(\nabla^i \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\nabla}^j \mathcal{D}^i \mathcal{E}_{ij}) \\ &\quad + \tfrac{1}{4} g^{ij} (\mathcal{D}^{[k} \mathcal{E}_{lj} \mathcal{D}^{l]} \mathcal{E}_{ki} + \bar{\mathcal{D}}^{[k} \mathcal{E}_{jl} \bar{\mathcal{D}}^{l]} \mathcal{E}_{ik}) \\ &\quad - \tfrac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{ij} \mathcal{D}_p \mathcal{E}_{kl} \\ &\quad - (\bar{\mathcal{D}}^j \mathcal{E}_{ij} \mathcal{D}^i d + \mathcal{D}^j \mathcal{E}_{ji} \bar{\mathcal{D}}^i d) - 4 \mathcal{D}^i d \, \mathcal{D}_i d \Big] \end{aligned}}$$

Surprising fact: dilaton EOM is

$$\mathcal{R}(\mathcal{E},d) = 0 \quad !!$$

$\nabla_i, \bar{\nabla}_i$ are $O(d,d)$ covariant derivatives

In the familiar action:

$$\int dx\, e^{-2\phi}\sqrt{-g}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right) \quad \text{---} \quad (A)$$

the dilaton EOM is

$$R + 4\left(\nabla^2\phi - (\partial\phi)^2\right) - \frac{1}{12}H^2 = 0$$

But in the alternative action

$$\int dx\, e^{-2\phi}\sqrt{-g}\left(R + 4\left(\nabla^2\phi - (\partial\phi)^2\right) - \frac{1}{12}H^2\right)$$

which differs from (A) by a total derivative

dilaton EOM is

$$R + 4\left(\nabla^2\phi - (\partial\phi)^2\right) - \frac{1}{12}H^2 = 0$$

So, we have:

$$\mathcal{R}(\mathcal{E}, d)\Big|_{\tilde{\partial}=0} = R + 4\left(\nabla^2\phi - (\partial\phi)^2\right) - \frac{1}{12}H^2$$

> this is the <u>curvature scalar</u>; the unique $O(D,D)$ invariant generalization of the scalar curvature R

Open questions

* Better understanding of the curvature scalar. Is there a Ricci, a Riemann.

* Compare with Siegel ($e_A{}^M, \ldots$)

* Better understanding of connections
  $O(D,D)$ vs. gauge invariance.

* Continue the higher order construction without the $\partial^M A \, \partial_M B = 0$ constraint.