M5-branes, 4d gauge theories, and 2d CFTs

Yuji Tachikawa (IAS)

based on works in collaboration with

L. F. Alday, F. Benini, S. Benvenuti,
D. Gaiotto, S. Gukov, S. Kanno,
Y. Matsuo, S. Shiba, H. Verlinde, B. Wecht

and discussions with many others

February 2010

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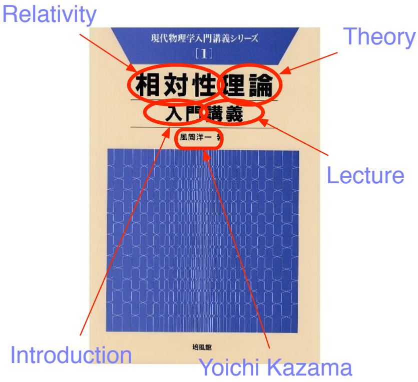

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10 heavenly stems

甲	乙	丙	丁	戊	己
tree↑	tree↓	fire↑	fire↓	earth↑	earth↓
庚	辛	壬	癸		
metal↑	metal↓	water↑	water↓		

12 earthly branches

子	丑	寅	卯	辰	巳
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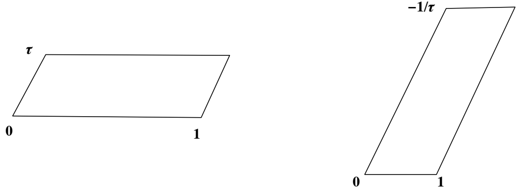
Happy Return of the Calendar!
還暦おめでとうございます

$\mathcal{N} = 4$ and M5-branes

- N M5-branes on S^1 $\longrightarrow$ N D4-branes: $\mathbf{SU}(N)$ SYM in 5d
- N M5-branes on T^2 $\longrightarrow$ $\mathbf{SU}(N)$ SYM in 4d
- 4d gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

is the shape of the torus



$\longrightarrow$ S-duality!

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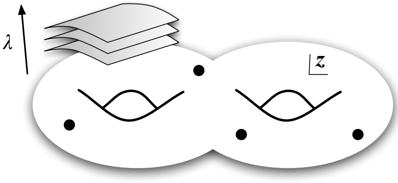
- I guess everyone was busy with AdS/CFT.

Rules

- Wrap N M5-branes on a Riemann surface with punctures.



- The worldvolume is given by

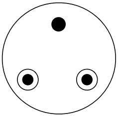
$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

- $\lambda = y dz$ is a differential. $\phi_k(z) = \varphi_k(z) dz^k$.
- Punctures determine the poles of $\phi_k(z)$.
- This is the Seiberg-Witten curve. (all this was known 12 years ago!)

More rules

- a tube gives $\mathbf{SU}(N)$ gauge group

- a three-punctured sphere gives the matter field e.g.

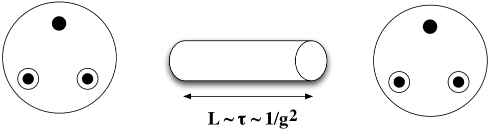


gives $N \times N$ hypermultiplets.

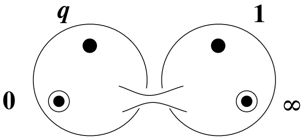
- Punctures carry flavor symmetries. $\odot$: $\mathbf{SU}(N)$ and $\bullet$: $\mathbf{U}(1)$

Practice!

- Want to make a $\mathbf{SU}(N)$ with $\mathbf{2}N$ fundamentals?
- Take N fundamental hypers, gauge fields, another N fundamental hypers



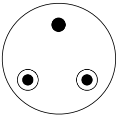
- Connect!



- $q \sim \exp(i\tau)$.

More practice

- $\mathbf{2} \times \mathbf{2}$ of $\mathbf{SU(2)} \times \mathbf{SU(2)}$



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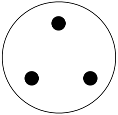

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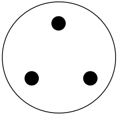

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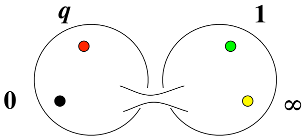
- $\mathbf{SU(2)}$ with four flavor is, then,



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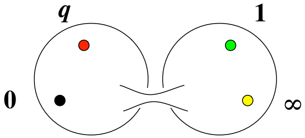

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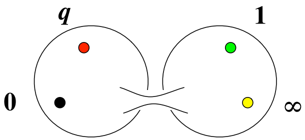

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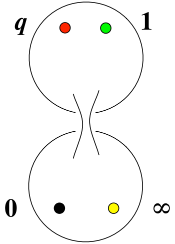

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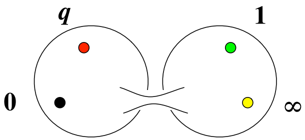

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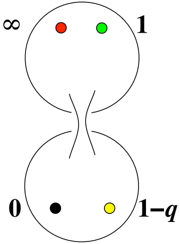

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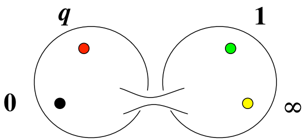

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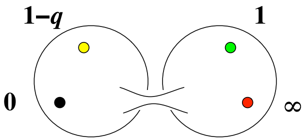

- this theory with coupling $q =$ the same with coupling $1 - q$.
- Hypers are in

$$\mathbf{2} \otimes \mathbf{2} \oplus \mathbf{2} \otimes \mathbf{2} \longleftrightarrow \mathbf{2} \otimes \mathbf{2} \oplus \mathbf{2} \otimes \mathbf{2}$$

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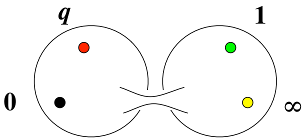

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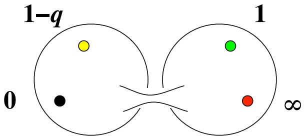

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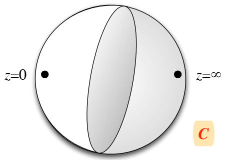
$$\text{vector of } \mathbf{SO(8)} \longleftrightarrow \text{spinor of } \mathbf{SO(8)}$$

2d CFT?

- All these pictures remind us of 2d CFTs. Pure $\mathbf{SU(2)}$:



- Looks like $\langle w|w\rangle$.
- Let $c = 1 + 6Q^2$, $Q = b + 1/b$. Fix $|w\rangle$ via

$$L_0|w\rangle = (Q^2/4 - a^2)|w\rangle, \qquad L_1|w\rangle = \Lambda^2|w\rangle.$$

Then

$$\langle w|w\rangle = 1 + \frac{2\Lambda^4}{(b+1/b)^2 - 4a^2} + \cdots$$

4d gauge theory?

- $d = 4\ \mathcal{N} = 2$ theory has Nekrasov's partition function

$$Z(a, \epsilon_1, \epsilon_2) = \exp\left(\frac{\mathcal{F}(a)}{\epsilon_1 \epsilon_2} + \cdots\right)$$

  which is explicitly calculable. $\mathcal{F}(a)$ is the prepotential.
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(proved by [Fateev-Litvinov,0912.0504])

4d theory vs 2d CFT

- We now have a map

$$G_N : \text{ Riemann surface with punctures} \longrightarrow \text{4d field theory}$$

- G_N behaves nicely under degenerations of the Riemann surface $\boldsymbol{\Sigma}$
  i.e. any thin, long tube gives a weakly coupled $\mathbf{SU}(N)$ gauge group

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- Take whatever physical quantity Z calculable in 4d:

$$Z : \text{ 4d field theory} \longrightarrow \text{ number}$$

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- Then, $Z(G_N(\Sigma))$ behaves nicely under degenerations of Σ,
- This morally means that $Z \circ G_N$ gives a 2d CFT.

6d CFT

- More physically, consider N M5-branes wrapped on $\boldsymbol{X_4 \times \Sigma_2}$
- Take a quantity Z for the 6d theory,
- furthermore suppose Z depends only on the complex structure.

$$\begin{tikzcd}
& \textbf{6d theory on } X_4 \times \Sigma_2 \arrow[dl, "\substack{\text{KK-reduce} \\ \text{along } \Sigma_2}"'] \arrow[dr, "\substack{\text{KK-reduce} \\ \text{along } X_4}"] & \\
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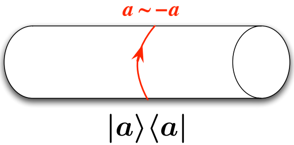
SU(2) vs. Liouville

- Take 4d quantities:
  prepotential $\mathcal{F}(a)$ or its generalization $Z_{\text{Nekrasov}}(a, \epsilon_1, \epsilon_2)$
- What do we get from 2 M5-branes?



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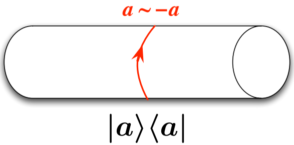

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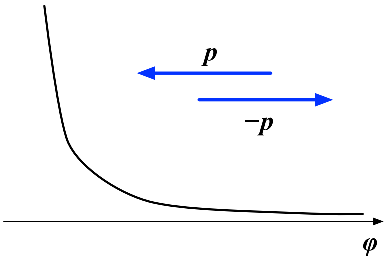
- Such 2d CFT is forced to be Liouville [Teschner,...]

(N.B. I learned this argument from Ari Pakman.)

Liouville

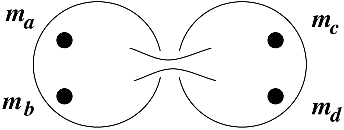


- Reflection off an exponential wall.
- The action is

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g} \left(|\partial_\mu \varphi|^2 + \mu e^{2b\varphi} + QR\varphi \right)$$

- $Q = b + 1/b$ and $c = 1 + 6Q^2$.

Proposal



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- Liouville theory has $T(z)$.
- Both have spin-2.

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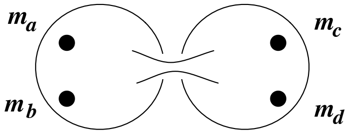

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where b^2 in Liouville = ϵ_1/ϵ_2 in gauge theory

- $\phi_2(z) \sim \dfrac{m^2 dz^2}{(z-z_i)^2}$ $\longrightarrow$ insertion of $e^{m\varphi(z_i)}$ in Liouville.

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Liouville theory is
the **quantization** of
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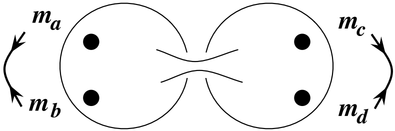
Pure SU(2)

- Take the limit $m_{1,2,3,4} \to \infty$, $g_{\mathrm{UV}} \to 0$ keeping Λ fixed:



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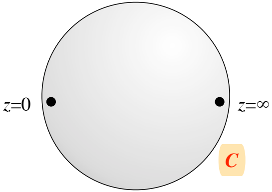

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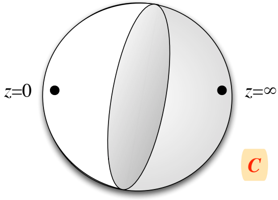

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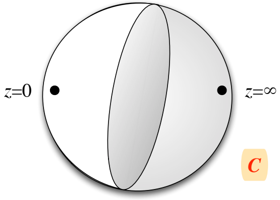

- $\langle w|w\rangle$. What is $|w\rangle$?
- The curve is

$$\Lambda^2(z + \frac{1}{z}) = y^2 - u$$

or equivalently

$$\left(y\frac{dz}{z}\right)^2 = \left(u + \Lambda^2(z + \frac{1}{z})\right)\left(\frac{dz}{z}\right)^2$$

Pure SU(2)

- Thus
$$\phi_2(z) \sim (\frac{u}{z^2} + \frac{\Lambda^2}{z^3})dz^2 \qquad \text{around} \quad z = 0$$
- Recall
$$T(z)dz^2 \sim (\cdots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \cdots)dz^2.$$

  this suggests
$$\begin{aligned} L_0|w\rangle &= a^2|w\rangle, \\ L_1|w\rangle &= \Lambda^2|w\rangle, \\ L_n|w\rangle &= 0 \qquad\qquad\qquad (n \geq 2). \end{aligned}$$
- $|w\rangle$ is the coherent state of the Virasoro algebra !
- and indeed, Nerkasov's $Z_{\text{inst}} = \langle w|w\rangle$.

Summary

- Systematic construction of 4d theories
  by putting M5-branes on a Riemann surface.

- very presice mapping between

  $$\text{4d quantity} \leftrightarrow \text{2d quantity}$$

  was accidentally found.

- conceptual understanding?