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Lattice formulation of supersymmetric gauge theories with exact supersymmetry

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1 Introduction

$\diamondsuit$ Lattice formulations are conventional formulations enable to access to nonperturbative aspects of quantum field theory.

$\diamondsuit$ We wish to extend such lattice study to supersymmetric (gauge) theories.
Difficulty for realization of SUSY on lattice
In general, $(\text{SUSY})^2 \sim (\underline{\text{infinitesimal translation}})$

$\uparrow$

Not a symmetry of lattice

$\diamondsuit$ However, a part of supercharges can be preserved on the lattice.
e.g.) Nilpotent supercharges (up to internal symmetry) do not induce translations.

($\Leftrightarrow$ scalar supercharges from topological twist)

Example: 2D $\mathcal{N} = (2,2)$ SYM or SQCD case:

Lattice formulation preserving the supercharge: $Q = -\frac{1}{\sqrt{2}}(Q_L + \bar{Q}_R)$
(BRST charge in A-model twist)

[Two R-symmetries]	[on lattice]
U(1)$_A$	O.K.
U(1)$_V$	broken

Another possibility preserving the supercharge: $Q' = -\frac{1}{\sqrt{2}}(\bar{Q}_L + \bar{Q}_R)$
(BRST charge in B-model twist)

[Two R-symmetries]	[on lattice]
U(1)$_A$	broken
U(1)$_V$	O.K.

- $\mathbf{U(1)}_V \Leftarrow$ chiral symmetry of 4D $\mathcal{N} = 1$
- $\mathbf{U(1)}_A \Leftarrow$ rotational symmetry on reduced 2D plane

Because we consider the theory on flat space-time, topological twists are just renaming the field variables of the continuum theory.

$\Rightarrow$ The continuum theory does not change by the twists.

However, the lattice theory becomes different depending on which of Q and Q' is exactly preserved.

◇ Plan of Talk

§ 1: Introduction

§ 2: Continuum 2D $\boldsymbol{\mathcal{N}=(2,2)}$ SYM

§ 3: Lattice formulation of 2D $\boldsymbol{\mathcal{N}=(2,2)}$ SYM

§ 4: 2D $\boldsymbol{\mathcal{N}=(2,2)}$ SQCD

§ 5: 2D $\boldsymbol{\mathcal{N}=(4,4)}$ SYM

§ 6: 2D lattice for 4D SYM

§ 7: Summary and Discussion

2 Continuum 2D $\mathcal{N} = (2,2)$ SYM

After taking the Wess-Zumino gauge,

◇ Euclidean Action

$$S^{(E)}_{(2,2)} = \frac{1}{g^2}\int d^2x \,\mathrm{tr}\Big(H^2 - 2iHF_{12} + D_\mu\phi D_\mu\bar{\phi} + \frac{1}{4}[\phi,\bar{\phi}]^2$$
$$+4\bar{\lambda}_R D_z \lambda_R + 4\bar{\lambda}_L D_{\bar{z}}\lambda_L + 2\bar{\lambda}_R[\bar{\phi},\lambda_L] + 2\bar{\lambda}_L[\phi,\lambda_R]\Big),$$

◇ Q-SUSY

$$\begin{aligned} &QA_\mu = \psi_\mu, \qquad Q\psi_\mu = iD_\mu\phi, \\ &Q\phi = 0, \\ &Q\bar{\phi} = \eta, \qquad Q\eta = [\phi,\bar{\phi}], \\ &Q\chi = H, \qquad QH = [\phi,\chi], \end{aligned}$$

$Q^2 =$ (infinitesimal gauge transformation with the parameter ϕ)

with the gaugino fields renamed as $\mathrm{U}(1)_A$

$$\psi_1 \equiv \frac{1}{\sqrt{2}}(\lambda_L + \bar{\lambda}_R), \qquad \psi_2 \equiv \frac{i}{\sqrt{2}}(\lambda_L - \bar{\lambda}_R), \qquad 1$$

$$\chi \equiv \frac{1}{\sqrt{2}}(\lambda_R - \bar{\lambda}_L), \qquad \eta \equiv -i\sqrt{2}(\lambda_R + \bar{\lambda}_L). \qquad -1$$

$\Rightarrow$The action can be expressed as the Q-exact form: [Witten]

$$S^{(E)}_{(2,2)} = Q\,\frac{1}{g^2_{2d}}\int d^2x\,\mathrm{tr}\left[\chi(-2iF_{12} + H) + \frac{1}{4}\eta[\phi, \bar{\phi}] - i\psi_\mu \mathcal{D}_\mu \bar{\phi}\right].$$

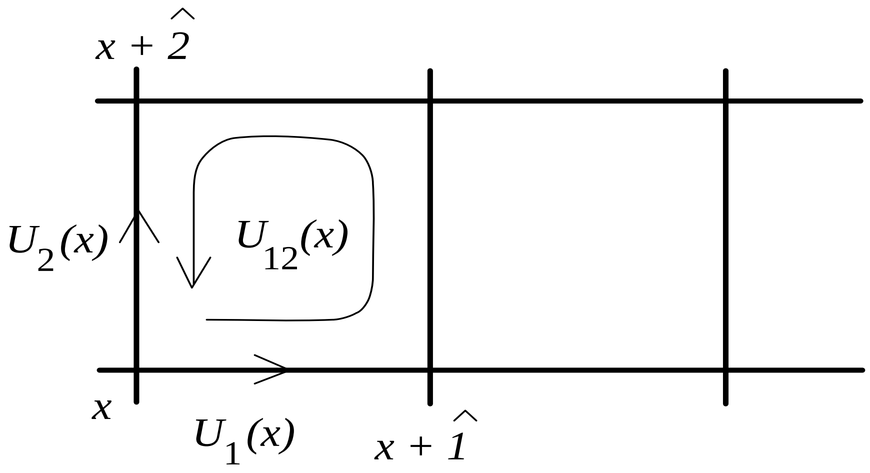

Figure 1: Link variables $U_\mu(x)$ and plaquette field $U_{12}(x)$. $U_{21}(x) = U_{12}(x)^{-1}$.

3 Lattice formulation of 2D $\mathcal{N} = (2,2)$ SYM

Lattice gauge fields are on links: $A_\mu(x) \Rightarrow U_\mu(x) = e^{iaA_\mu(x)}$
All the other fields are on sites.

Lattice fields are dimensionless.

$$\phi, \bar\phi = \mathcal{O}(a), \qquad \psi_\mu, \chi, \eta = \mathcal{O}(a^{3/2}), \qquad H = \mathcal{O}(a^2),$$
$$Q = \mathcal{O}(a^{1/2}).$$

◇ Q-SUSY on the lattice

$$
\begin{aligned}
QU_\mu(x) &= i\psi_\mu(x)U_\mu(x), \\
Q\psi_\mu(x) &= i\psi_\mu(x)\psi_\mu(x) + ia\nabla_\mu\phi(x), \\
Q\phi(x) &= 0, \\
Q\bar{\phi}(x) &= \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)], \\
Q\chi(x) &= H(x), \quad QH(x) = [\phi(x), \chi(x)],
\end{aligned}
$$

where $a\nabla_\mu\phi(x) \equiv U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^{-1} - \phi(x)$.

$\Rightarrow Q^2 =$ (infinitesimal gauge tr. with the parameter $\phi(x)$)
(Nilpotent up to gauge transformation on the lattice)

- Actually, starting with $QU_\mu(x) = i\psi_\mu(x)U_\mu(x)$, we get $Q\psi_\mu(x)$ as

$$
\underline{Q^2U_\mu(x)} = i(Q\psi_\mu(x))U_\mu(x) - i\psi_\mu(x)\left(QU_\mu(x)\right)
$$

$$
\Downarrow
$$

$$
\phi(x)U_\mu(x) - U_\mu(x)\phi(x+\hat{\mu})
$$

◇ Q-invariant Lattice Action: Q(gauge invariant terms) [F.S.]

For admissible gauge fields ($||1 - U_{12}(x)|| < \epsilon$ for $\forall x$),

$$S^{(\mathrm{lat})}_{(2,2)} = Q\frac{1}{g_0^2}\sum_x \mathrm{tr}\Big[\chi(x)\left\{-i\widehat{\Phi}(x) + H(x)\right\} + \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\sum_\mu \psi_\mu(x) a\nabla_\mu\bar{\phi}(x)\Big],$$

Otherwises, $S^{(\mathrm{lat})}_{\mathrm{SYM}} = +\infty$. (i.e. The Boltzmann weight is zero.)

Here, (for $G = \mathrm{U}(N)$)

$$\widehat{\Phi}(x) = \frac{-i(U_{12}(x) - U_{21}(x))}{1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2} \sim 2F_{12}$$

Note

If we used a naive $-i(U_{12}(x) - U_{21}(x))$ instead of $\widehat{\Phi}$,

gauge kinetic terms would be

$$\sim -\mathrm{tr}\,(U_{12}(x) - U_{21}(x))^2 = \mathrm{tr}\,(2 - U_{12}(x)^2 - U_{21}(x)^2)$$

$\Rightarrow$ The configurations

$$U_{12}(x) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \qquad \text{(up to gauge tr.)}$$

for $\forall x$ would give the classical minima of the action.

Huge degeneracy! ($\sharp$ of minima) $\sim \mathcal{O}\left(2^{N(\sharp \text{ of plaquettes})}\right)$

Because the continuum theory is derived from weak field expansion around $U_\mu(x) = 1$, we should single out the vacuum $U_{12}(x) = 1$.

$\Rightarrow$ The use of $\widehat{\Phi}$ does the job with keeping Q-SUSY.

c.f.) The Wilson lattice gauge action: $\mathbf{tr}\,(2 - U_{12}(x) - U_{21}(x))$
$\Rightarrow$ The unique minimum $U_{12}(x) = 1$.

Note:

- ϵ is some positive number independent of the lattice spacing a.
  $\Rightarrow$ F_{12} is almost unconstrained near the continuum limit.
- O.K. for $G = \mathrm{U}(N), \mathrm{SU}(N)$.

◇ Restoration of full SUSY

Of course, the lattice action reduces to the continuum classical action in the continuum limit $a \longrightarrow 0$ with $g_{2d} \equiv g_0/a$ fixed.

How about quantum mechanically?

Possible relevant/marginal operators radiatively generated:

- $\mathrm{tr}\,\phi,\quad \mathrm{tr}\,\bar{\phi},\quad \mathrm{tr}\,H$: forbidden by $\mathrm{U}(1)_A$ symmetry and reflection symmetry of the lattice action
- $\mathrm{tr}\,\phi\bar{\phi}$: forbidden by Q-SUSY.

There does not appear dangerous operator which prevents from restoring full SUSY. $\Rightarrow$ No fine tuning needed

This is a perturbative argument.

Nonperturbative check for the restoration has been done by numerical study in case of $G = \mathrm{SU}(2)$. [Kanamori-Suzuki]

4 Continuum 2D $\mathcal{N}=(2,2)$ SQCD

2D $\mathcal{N}=(2,2)$ SQCD $\Leftarrow$ dimensional reduction from 4D $\mathcal{N}=1$ SQCD

Field contents:

- $V=(A_\mu,\phi,\bar{\phi};\lambda;H) \Leftarrow$ 4D $\mathcal{N}=1$ vector superfield
- $\Phi_{+I}=(\phi_{+I};\psi_{+IR},\psi_{+IL};F_{+I}) \Leftarrow$ 4D $\mathcal{N}=1$ chiral superfield
  (fundamental repre., flavors: $I=1,\cdots,n_+$)
- $\Phi_{-I}=(\phi_{-I};\psi_{-IR},\psi_{-IL};F_{-I}) \Leftarrow$ 4D $\mathcal{N}=1$ chiral superfield
  (anti-fundamental repre., flavors: $I=1,\cdots,n_-$)

The continuum Lagrangian :

$$\mathcal{L} = \mathcal{L}_{\mathrm{SYM}} + \left[\sum_{I=1}^{n_+}\Phi^\dagger_{+I}e^V\Phi_{+I} + \sum_{I=1}^{n_-}\Phi_{-I}e^{-V}\Phi^\dagger_{-I}\right]\Big|_{\theta\theta\bar{\theta}\bar{\theta}}$$
$$+ W(\Phi_+,\Phi_-)|_{\theta\theta} + \overline{W}(\Phi^\dagger_+,\Phi^\dagger_-)\Big|_{\bar{\theta}\bar{\theta}} ,$$

$\diamondsuit$ Lattice formulation of 2D $\mathcal{N} = (2, 2)$ SQCD

Ginsparg-Wilson formulation can be extended to realize both of Q-SUSY and chiral flavor symmetry. $\longrightarrow$ O.K. for general $n_{\pm}$ [Kikukawa-F.S.]

- O.K. for general superpotential with preserving holomorphic structure
- Application to gauged linear sigma models

$\diamondsuit$ Q' invariant lattice formulation (B-model twist): [Kadoh-F.S.-Suzuki]

seems more natural to preserve the chiral flavor symmetry.

- In the case $n_- = 0$ (only fund.) or $n_+ = 0$ (only anti-fund.), the constructed action can be shown to have the single minimum $U_{12}(x) = 1$ without using the admissibility conditions.
- $G = \mathrm{U}(N)$

Superpotentials:

- $\overline{W}$ is Q'-exact. $\Rightarrow$ $\overline{W}$ is Q'-invariant on the lattice.
- W is not Q'- exact. $\Rightarrow$ difficult to realize Q'-invariant W on the lattice.

5 2D $\mathcal{N} = (4,4)$ SYM

2D $\mathcal{N} = (4,4)$ SYM $\Leftarrow$ (dim. red.) $=$ 4D $\mathcal{N} = 2$ SYM

Bosons : $A_\mu, \quad B, \quad C, \quad \phi, \quad \bar{\phi}$
Fermions : $\psi_{\pm\mu}, \quad \chi_\pm, \quad \eta_\pm$

There are two supercharges $Q_\pm$ nilpotent in the sense of

$$\begin{aligned} Q_+^2 &= (\text{infinitesimal gauge transformation with parameter } \phi), \\ Q_-^2 &= (\text{infinitesimal gauge transformation with parameter } -\bar{\phi}), \\ \{Q_+, Q_-\} &= (\text{infinitesimal gauge transformation with parameter } C). \end{aligned}$$

Under $\mathrm{SU}(2)_R$ subgroup of the R-symmetry group $\mathrm{SU}(4)$,

$(Q_+, Q_-), \quad (\psi_+, \psi_-), \quad (\chi_+, \chi_-), \quad (\eta_+, -\eta_-)$: doublets,
$(\phi, C, \bar{\phi})$: triplet.

The classical action can be expressed as the $Q_\pm$ exact form: [Dijkgraaf-Moore], [Blau-Thompson],..

$$
\begin{aligned}
S^{(E)}_{(4,4)} &= Q_+ Q_- \mathcal{F}_{(4,4)}, \\
\mathcal{F}_{(4,4)} &\equiv \frac{1}{g_{2d}^2} \int d^2x \, \mathrm{tr} \left[-2iBF_{01} - \psi_{+\mu}\psi_{-\mu} - \chi_+\chi_- - \frac{1}{4}\eta_+\eta_- \right].
\end{aligned}
$$

$\mathcal{F}_{(4,4)}$: gauge and $\mathrm{SU}(2)_R$ invariant.

The lattice action with $Q_\pm$-SUSY can be constructed essentially by $2F_{12} \longrightarrow \widehat{\Phi}(x)$ other than trivial changes. [F. S.]

Perturbatively, it is shown that full SUSY is restored in the continuum limit with no fine-tuning.

6 2D lattice for 4D SYM

◇ Mass-deformed 2D $\mathcal{N} = (4,4)$ SYM

[Hanada-Matsuura-F.S.] (in progress)

We can consider a mass-deformation analogous to the BMN matrix model to the 2D $\mathcal{N} = (4,4)$ action with keeping the full 8 SUSY as

$$S^{(E)}_{(4,4)M} = \frac{2}{g_{2d}^2}\int d^2x \, \mathrm{tr}\Big[\frac{1}{2}F_{12}^2 + \frac{1}{2}(D_\mu X^I)^2 + \frac{1}{2}\Psi^T(D_1 + \gamma_2 D_2)\Psi + \frac{i}{2}\Psi^T\gamma_I[X^I,\Psi] - \frac{1}{4}[X^I,X^J]^2 + \frac{1}{2}\left(\frac{M}{3}\right)^2(X^a)^2 - i\frac{M}{6}\Psi^T\gamma_{23}\Psi + i\frac{M}{3}X^3F_{12} + i\frac{M}{3}\epsilon_{abc}X^aX^bX^c\Big]$$

with $I,J = 3,4,5,6$ and $a,b,c = 4,5,6$. c.f. [Das-Michelson-Shapere]

Then, $Q_\pm$-SUSY is deformed so that they are nilpotent up to gauge and $\mathrm{SU}(2)_R$ rotations :

$$Q_+^2 = (\text{gauge transf. with } \phi) + \frac{M}{3}J_{++},$$

$$\begin{aligned} Q_-^2 &= (\text{gauge transf. with } -\bar{\phi}) - \frac{M}{3} J_{--}, \\ \{Q_+, Q_-\} &= (\text{gauge transf. with } C) - \frac{M}{3} J_0 \end{aligned}$$

with $J_0, J_{\pm\pm}$: generators of $\mathrm{SU}(2)_R$.

The action is expressed as

$$S^{(E)}_{(4,4)M} = \left(Q_+ Q_- - \frac{M}{3}\right) \mathcal{F}_{(4,4)},$$

where $\mathcal{F}_{(4,4)}$ is identical with the undeformed case.

Note that $S^{(E)}_{(4,4)M}$ is not precisely Q_+Q_--exact but $Q_\pm$ invariant.
For instance,

$$\begin{aligned} Q_+ S^{(E)}_{(4,4)M} &= Q_+^2 Q_- \mathcal{F}_{(4,4)} - \frac{M}{3} Q_+ \mathcal{F}_{(4,4)} \\ &= \frac{M}{3} J_{++} Q_- \mathcal{F}_{(4,4)} - \frac{M}{3} Q_+ \mathcal{F}_{(4,4)} = 0. \end{aligned}$$

$\diamondsuit$ $Q_\pm$ invariant lattice action for the deformed theory can be obtained again by $2F_{12} \longrightarrow \widehat{\Phi}(x)$.

Perturbatively, full SUSY is restored in the continuum limit without fine-tuning.

• The theory has no flat directions, but discrete minima corresponding to Fuzzy spheres

$$[X^a, X^b] = i\frac{M}{3}\epsilon_{abc}X^c.$$

Fuzzy sphere configurations are supersymmetric.

$\Rightarrow$ Expanding around one of the configurations

$$X^a = \frac{M}{3}L^a_{(k)} \otimes \mathbb{1}_n, \qquad (L^a_{(k)} : k\text{-dim. irre. rep. of SU(2)})$$

we obtain 4D $\mathcal{N} = 2$ SYM on $R^2\times$ Fuzzy S^2.

- gauge group of the 4d theory : $G_{4d} = \mathrm{U}(n)$ (n finite!)
- "lattice spacing" of Fuzzy S^2 : $b = 1/(Mk)$,

NC parameter $\theta \sim 1/(Mk^2)$
$b \longrightarrow 0$ ($k \longrightarrow \infty$) limit is smooth? UV-IR mixing?

◇ We are considering the similar thing for mass-deformed 2d $\mathcal{N} = (8, 8)$ SYM. [Hanada-Matsuura-F.S.] (in progress)

- The continuum 2d theory would be obtained from the lattice theory with no fine-tuning.
- 4D $\mathcal{N} = 4$ SYM on $R^2 \times$ Fuzzy S^2 is obtained.
  Now, $b \longrightarrow 0$ ($k \longrightarrow \infty$) limit is expected to be smooth.
  $\Rightarrow$ Continuum 4d $\mathcal{N} = 4$ SYM with $G_{4d} = \mathrm{U}(n)$ on $R^2 \times S^2$ could be obtained with no fine-tuning. ($1/M$: radius of S^2)
- Finally, take $M \longrightarrow 0$ limit to get the theory on R^4.

$\Rightarrow$ If this scenario works, the 2D lattice would give the nonperturbative formulation of 4d $\mathcal{N} = 4$ SYM with $G_{4d} = \mathrm{U}(n)$.

7 Summary and Discussion

$\diamondsuit$ Lattice formulations of 2D $\mathcal{N} = (2, 2)$ SYM (and SQCD) have been discussed.

- Q-invariant formulation (A-model twist)
  - $G = \mathrm{U}(N), \mathrm{SU}(N)$
  - general $n_\pm$ by employing the Ginsparg-Wilson operator
    Exact chiral flavor symmetry on the lattice
  - general superpotentials
- Q'-invariant formulation (B-model twist)
  simpler than the A-model twist case (Admissibility cond. is not necessary)
  - $G = \mathrm{U}(N)$
  - $(n_+, n_-) = (n_+, 0)$ or $(0, n_-)$

$\diamondsuit$ Lattice fomulation of 2D $\mathcal{N} = (4, 4)$ SYM and its mass-deformed version has been discussed.

- Hybrid formulation of 2D lattice and Fuzzy S^2 seems to be an interesting possibility to construct 4D $\mathcal{N} = 4$ SYM.
  - Interesting also from the viewpoint of gauge/string duality.

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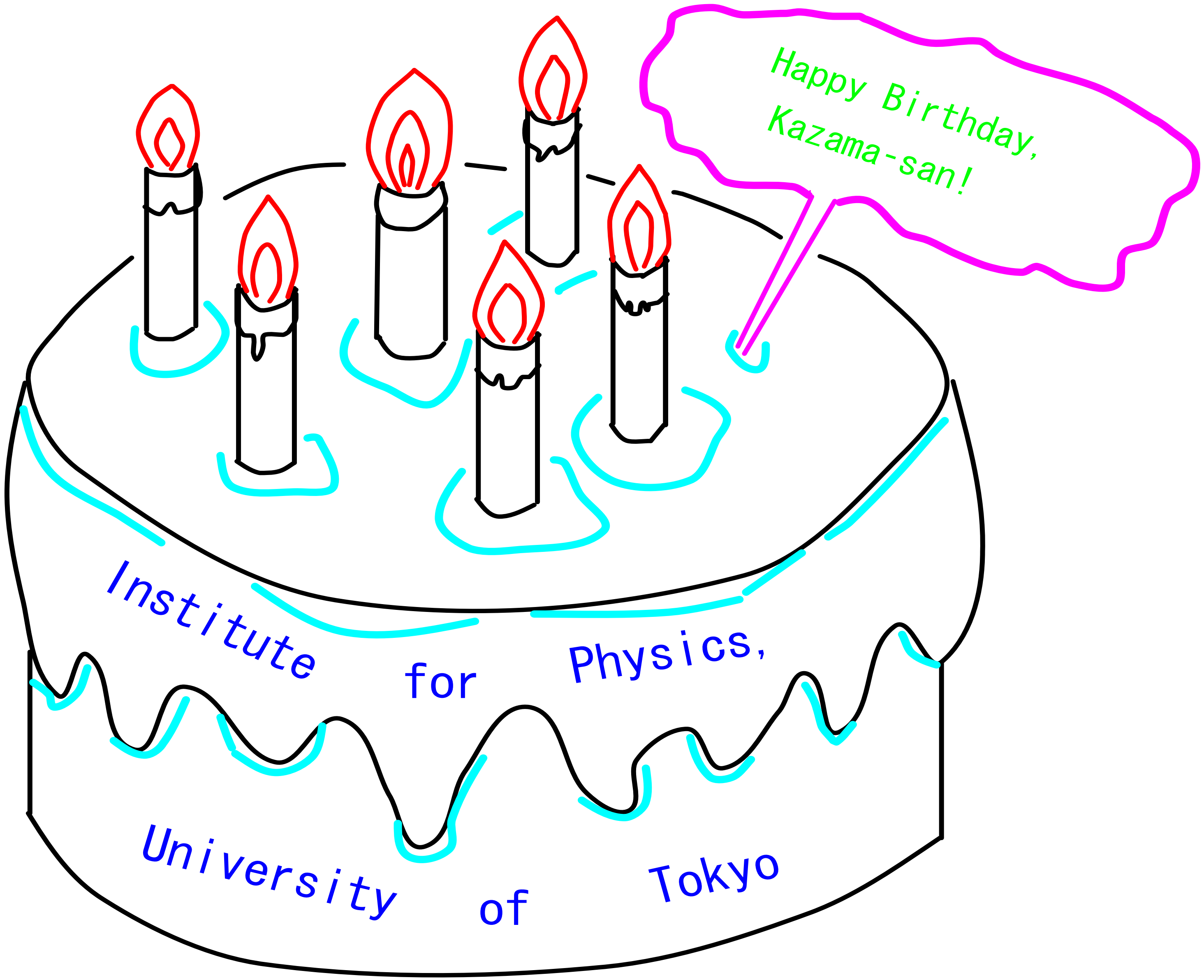
  - Interesting also from the viewpoint of gauge/string duality.

Thank you very much!

Happy Birthday, Kazama-san!


Happy Birthday,
Kazama-san!
Institute for Physics,
University of Tokyo