An Overture to Quantum Superstring in $AdS_5 \times S^5$

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1 Introduction

The concept of AdS/CFT (or gauge/string) correspondence has had enormous impact on both string theory and field theory.

It has been applied to a variety of problems in diverse areas.

Holographic QCD, AdS/CMT, etc.

($\sim$ strong coupling problems in field theory)

But

the fundamental understanding of this correspondence still remains as an important unsolved problem.

(My main interest in the past 10 years)

Major reason for the difficulty:
strong/weak nature of the correspondence

For $N = 4$ SYM / SG in $AdS_5 \times S^5$ correspondence

$$\boxed{(\star) \quad g_{YM}^2 N = 4\pi g_s N = \frac{R^4}{\alpha'^2}}$$

Small (large) 't Hooft coupling $\Longleftrightarrow$ large (small) $\alpha' \sim l_s^2$

Two equalities in the relation

- First equality : Perturbative open-closed duality
- Second equality (closed string side only)
  $\sim$ Balance between gravity and RR flux ("anti-gravity")

Lagrangian density

$$\mathcal{L}_{gravity} \sim \frac{1}{g_s^2}\frac{1}{l_s^8}\mathcal{R} \sim \frac{1}{g_s^2}\frac{1}{l_s^8}\frac{1}{R^2} = \mathcal{L}_{RR} \sim F_5^2 \sim \left(\frac{N}{R^5}\right)^2$$

Clearly, it is not just an open-closed duality

- AdS side: Do need stringy corrections ($\Leftarrow$ BMN)
- **CFT** side: No need of contributions from massive modes of the open string

Why is the (supergravity level) relation $(\star)$ not corrected ?

For deeper understanding, we wish to solve
the quantum closed string in AdS spacetime
either exactly or in $1/\alpha'(\sim \sqrt{\lambda})$ expansion

Past studies:

☐ **Quantum string in AdS_3 with NSNS flux** :

- RNS formalism with worldsheet conformal invariance

  (Giveon-Kutasov-Seiberg (98), Maldacena-Ooguri (00,01), etc.)

  Bosonic part: $SL(2,R)$ WZW model

  – Isometry: $SO(2,2) = SL(2,R)_L \times SL(2,R)_R$

  $\longrightarrow$ current algebra symmetry $\widehat{SL(2,R)}_L \times \widehat{SL(2,R)}_R$

  – Spectrum and correlation functions are dictated by these powerful symmetries

♠ **Much more difficult with RR flux of our interest:**

RR field = bispinor: Difficult to handle in RNS formalism

Formalisms to handle RR fields

- ♦ Green-Schwarz formalism: LC (or Semi-LC) gauge (Green-Schwarz (84)
- ♦ Hybrid formalism = GS$\oplus$ RNS (Berkovits (94))
- ♦ Pure spinor formalism (Berkovits (00))

□ **Quantum string in AdS_3 (and AdS_2) with RR flux :**

▶ Conformal sigma model with a certain supergroup as a target space

(Berkovits-Vafa-Witten (99), Berkovits-Bershadsky-Hauer-Zhukov-Zwiebach (99), etc)

– Hybrid formalism: With RR flux, difficult to assure unitarity

□ **Quantum string in pp-wave background with RR flux**:

- Exact (free massive) spectrum using GS formalism in LC gauge (Metsaev (01))
- Correspondence with non-BPS operators in $\mathcal{N} = 4$ SYM

  (Berenstein-Maldacena-Nastase (02), many many works)
- Interaction: 3-point vertex in light-cone SFT

  (Spradlin, Volovich, . . . , Moriyama, . . . Dobashi, Shimada, Yoneya, . . .)

  Only partially understood in perturbation theory: not enough symmetry

- Worldsheet formulation with conformal invariance (in Semi-LC gauge) (YK and Yokoi (08), Comprehensive SLC gauge formulation (flat space) (YK and Yokoi, in preparation))

□ Quantum string in $AdS_5 \times S^5$ with RR flux:

Notoriously difficult problem

- Classical action: GS: (Metsaev-Tseytlin (98)) PS: (Berkovits (00)
- Classical GS LC gauge action and symmetry generators (Metsaev, Tseytlin, Thorn, (00, 01))
- $\alpha'(\sim 1/\sqrt{\lambda})$- expansions around classical solutions can be made.
- Attempts at formulation as an integrable model using Pohlmeyer reduction (Grigoriev, Tseytlin, Mikhailov, Shafer-Nameki, Roiban, Jevicki, ...)
  Classical solutions $\oplus$ semiclassical quantization
- So far, no formulation has been found where $psu(2,2|4)$ is lifted to current algebra symmetry.

♠ **Even a superparticle has not been fully quantized and solved in $AdS_5 \times S^5$**

Our Strategy

Maximal supersymmetry should be rather powerful

Make full use of the representation theory of
$psu(2,2|4)$ symmetry in the quantization

First stage: Spectrum: Which representations occur ?

1. Develop systematic phase space formulation from the first principle in a physical gauge and quantize the system.
2. Construct the quantum superconformal ($psu(2,2|4)$) generators in terms of string coordinates
3. Find all the superconformal primary states of the system by solving the equations $S_I|\Psi\rangle = 0 \Longrightarrow$ Complete spectrum

Each of these steps is non-trivial

Recently, we have been able to execute this program

for the zero mode part of the superstring (*i.e.* a superparticle)

T.Horigane and Y. Kazama arXiv:0912.1166 (Phys.Rev. D81, 045004, 2010)

Result

- **All the superconformal primary states and their explicit wave functions are obtained**
- **They precisely match the supergravity spectrum, including all the KK excitations**

Hopefully, this will serve as

an overture to quantum superstring in $AdS_5 \times S^5$

Plan

1. Introduction

2. Phase space formulation of a superparticle in $AdS_5 \times S^5$ with RR flux

3. Quantum superconformal generators

4. Complete solution of physical states
— Spectrum and wave functions —

5. On-going and future projects

2 Phase space formulation of a superparticle in $AdS_5 \times S^5$ with RR flux

2.1 Global symmetry algebra: $psu(2,2|4) \simeq$ superconformal algebra in 4D

The most efficient way to construct the Brink-Green-Schwarz action for a superparticle (and a superstring) in $AdS_5 \times S^5$ with RR flux is the supercoset construction based on the symmetry algebra $psu(2,2|4)$[1] .

$$AdS_5 \times S^5 \simeq \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)}$$

[1] Here we follow Metsaev and Tseytlin (hep-th/9805028, hep-th/0007036)

□ $psu(2,2|4)$ algebra:

Even part:

$SO(4,2)$

$$[T^{AB}, T^{CD}] = \eta^{BC} T^{AD} - \eta^{AC} T^{BD} - \eta^{BD} T^{AC} + \eta^{AD} T^{BC}$$

$$A = (-1, a, 4)\,, \quad a = 0,1,2,3\,, \quad \eta_{AB} = (-,-,+,+,+,+)$$

$SO(4,1)$ basis: $(\hat{a} = 0 \sim 4 = (a, 4))$

$$T^{AB} = \begin{cases} T^{\hat{a}\hat{b}} = SO(4,1) \text{ generators} \\ T^{\hat{a}} (\equiv T^{\hat{a},-1}) = \text{coset generators} \end{cases}$$

"4D-Conformal" basis: $(a = 0 \sim 3)$

$$P^a \equiv \frac{1}{\sqrt{2}}(T^a - T^{a4})\,, \qquad K^a \equiv -\frac{1}{\sqrt{2}}(T^a + T^{a4})$$

$$D \equiv T^4 (= T^{4,-1})\,, \qquad J^{ab} \equiv T^{ab}$$

Commutation relations

$$[\boldsymbol{J},\boldsymbol{P}]\sim\boldsymbol{P}\,,\qquad[\boldsymbol{J},\boldsymbol{K}]\sim\boldsymbol{K}\,,\qquad[\boldsymbol{J},\boldsymbol{J}]\sim\boldsymbol{J}$$

$$[\boldsymbol{D},\boldsymbol{P}]=\boldsymbol{P}\,,\quad[\boldsymbol{D},\boldsymbol{K}]=-\boldsymbol{K}\,,\qquad[\boldsymbol{P},\boldsymbol{K}]\sim\boldsymbol{D}\oplus\boldsymbol{J}$$

- We will often use the following 4D "Light-cone basis"

$$\boldsymbol{P}^{\pm}=\frac{1}{\sqrt{2}}(\boldsymbol{P}^{3}\pm\boldsymbol{P}^{0})$$

$$\boldsymbol{P}^{x}\equiv\frac{1}{\sqrt{2}}(\boldsymbol{P}^{1}+i\boldsymbol{P}^{2})\,,\quad\boldsymbol{P}^{\bar{x}}\equiv\frac{1}{\sqrt{2}}(\boldsymbol{P}^{1}-i\boldsymbol{P}^{2})$$

Similarly for $\boldsymbol{K}^{\pm}\,,\boldsymbol{K}^{x}\,,\boldsymbol{K}^{\bar{x}}\,,\boldsymbol{J}^{+-}\,,\boldsymbol{J}^{\pm x}\,,\boldsymbol{J}^{\pm\bar{x}}\,,\boldsymbol{J}^{x\bar{x}}$

$\boxed{\boldsymbol{SO}(6)\simeq\boldsymbol{SU}(4)}$

$$\left[\boldsymbol{J}^{i}{}_{j},\boldsymbol{J}^{k}{}_{l}\right]=\delta^{k}_{j}\boldsymbol{J}^{i}{}_{l}-\delta^{i}_{l}\boldsymbol{J}^{k}{}_{j}\,,\quad(\boldsymbol{J}^{i}{}_{i}=0)\qquad i,j,k,l=1\sim4$$

Odd part:

- 32 supercharges: $Q^{\pm i}\,, Q_i^{\pm}\,, S^{\pm i}\,, S_i^{\pm}\,, \quad i = 1 \sim 4$

Charges:

	$Q^{\pm i}$	$Q_i^{\pm}$	$S^{\pm i}$	$S_i^{\pm}$
D-charge	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$J^{\pm}$-charge	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$J^{x\bar{x}}$-charge	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$

$SU(4)$ transformation properties

$Q^{\pm i}, S^{\pm i}$ belong to 4, $\qquad$ $Q_i^{\pm}, S_i^{\pm}$ belong to $\bar{4}$

Non-vanishing $[\boldsymbol{B}_{oson}, \boldsymbol{F}_{ermion}]$ type commutators

$$[P, S] \sim Q\,, \qquad [K, Q] \sim S$$

$$[J, Q] \sim Q\,, \qquad [J, S] \sim S$$

- Anti-commutation relations among the supercharges:

$$\{\boldsymbol{Q},\boldsymbol{Q}\} \sim \boldsymbol{P}\,, \qquad \{\boldsymbol{S},\boldsymbol{S}\} \sim \boldsymbol{K}$$

$$\{\boldsymbol{Q},\boldsymbol{S}\} \sim \boldsymbol{J}_{\boldsymbol{so}(3,1)} \oplus \boldsymbol{D} \oplus \boldsymbol{J}_{\boldsymbol{su}(4)}$$

Hermiticity:

- Convention: $\boldsymbol{P}^{a}, \boldsymbol{K}^{a}, \boldsymbol{D}, \boldsymbol{J}^{ab}$ are anti-hermitian, $(\boldsymbol{J}^{i}{}_{j})^\dagger = \boldsymbol{J}^{j}{}_{i}$
- $(\boldsymbol{Q}^{\pm i})^\dagger = \boldsymbol{Q}_{i}^{\pm}$, $(\boldsymbol{S}^{\pm i})^\dagger = \boldsymbol{S}_{i}^{\pm}$

2.2 Supercoset construction

Supercoset

$$\mathcal{K} = \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

Our choice of the representative $G \in \mathcal{K}$

$$
\begin{aligned}
G &= g_x g_\theta g_\eta g_y g_\phi \\
g_x &= \exp(x_a P^a) \\
g_\theta &= \exp\left(\theta^{-i} Q_i^+ + \theta_i^- Q^{+i} + \theta^{+i} Q_i^- + \theta_i^+ Q^{-i}\right) \\
g_\eta &= \exp\left(\eta^{-i} S_i^+ + \eta_i^- S^{+i} + \eta^{+i} S_i^- + \eta_i^+ S^{-i}\right) \\
g_y &= \exp({y^i}_j {J^j}_i)\,, \qquad {y^i}_j = \frac{i}{2} \underbrace{{(\gamma^{A'})^i}_j}_{SO(5)\ \gamma} y^{A'}\,, \quad A' = 5 \sim 9 \\
g_\phi &= \exp(\phi D)
\end{aligned}
$$

10 bosonic coordinates $(x_a, \phi, y^{A'})$ and 32 fermionic coordinates $\theta^{\pm i}, \theta_i^{\pm}, \eta^{\pm i}, \eta_i^{\pm}$

Basic building blocks: **Left-invariant Cartan 1-forms**

$$J = G^{-1}dG = L_P^a P_a + L_K^a K_a + \cdots + L_S^{+i} S_i^- + \frac{1}{2} L^{ab} J_{ab} + L^j{}_i J^i{}_j$$

All the generators appear (unlike the flat case). $G^{-1}dG$ as a whole describes a motion in the entire group space.

Extract **the motion along the coset part** $\propto T^{\hat{a}}, J^{A'}$ $(\equiv -\frac{i}{2} J^i{}_j (\gamma^{A'})^j{}_i)$:
So we need to project it out like

$$\begin{aligned} J &= L_P^a P_a + L_K^a K_a + \cdots \\ &= L_P^a \frac{1}{\sqrt{2}} (T^a - T^{a4}) - L_K^a \frac{1}{\sqrt{2}} (T^a + T^{a4}) + \cdots \end{aligned}$$

$$\therefore \quad \begin{aligned} J_B^{coset} &= L_B dt = \frac{1}{\sqrt{2}} (L_P^a - L_K^a) T^a + \cdots \\ &= J_B^{\hat{a}} T_{\hat{a}} + J_B^{A'} J_{A'} \end{aligned}$$

$$J_B^{\hat{a}} = (J_B^a, J_B^4) = \left(\frac{1}{\sqrt{2}} (L_P^a - L_K^a), L_D \right), \quad J_B^{A'} = \frac{i}{2} (\gamma^{A'})^i{}_j L^j{}_i$$

Action

$$S = \frac{1}{2e} \int \left(\eta_{\hat{a}\hat{b}} J_B^{\hat{a}} J_B^{\hat{b}} + \delta_{A'B'} J_B^{A'} J_B^{B'} \right)$$

- For a superparticle the Wess-Zumino term vanishes. The local κ-symmetry already exists for S above.

□ Choice of a physical gauge :

Without imposing any gauge conditions, the Cartan 1-form $G^{-1}dG$ can contain up to 32 powers of fermionic coordinates. Impossible to compute it in closed form.

We fix the κ-symmetry by imposing the 16 conditions

$$\boxed{\theta^{+i} = \theta_i^+ = \eta^{+i} = \eta_i^+ = 0 \qquad \text{“light-cone gauge”}}$$

$\Leftrightarrow$ Only the supercharges with J^{+-}-charge $+\frac{1}{2}$ are kept

$$g_\theta = \exp\left(\theta^i Q_i^+ + \theta_i Q^{+i}\right)$$
$$g_\eta = \exp\left(\eta^i S_i^+ + \eta_i S^{+i}\right)\,, \qquad \theta^i \equiv \theta^{-i}\,, \eta^i \equiv \eta^{-i}\,,\ etc.$$

Multiple commutators terminate $\Rightarrow$ Cartan 1-form and the action simplify substantially.

$$S = \frac{1}{2e}\int dt \Bigg(\underbrace{e^{-2\phi}\left(\dot{x}^+\dot{x}^- + \dot{x}\dot{\bar{x}}\right) + (\dot{\phi})^2}_{AdS \text{ metric in Poincaré coord.}(z = e^{\phi})} + e_0^{A'} e_0^{A'}$$
$$- \frac{i}{2}\dot{x}^+\left[e^{-2\phi}(\theta^i\dot{\theta}_i + \theta_i\dot{\theta}^i) + (\eta^i\dot{\eta}_i + \eta_i\dot{\eta}^i) - 2i\tilde{\eta}_i(\gamma^{A'})^i{}_j e_0^{A'}\tilde{\eta}^j\right]$$
$$- \frac{1}{4}(\dot{x}^+)^2\left[(\eta^2)^2 - (\tilde{\eta}_i(\gamma^{A'})^i{}_j\tilde{\eta}^j)^2\right]\Bigg)$$

where

$$U = \exp((i/2)y^{A'}\gamma^{A'})\,, \quad e_0^{A'} = -\frac{i}{2}\mathrm{Tr}\,(\gamma^{A'}\dot{U}U^{-1})\,, \quad \tilde{\eta}^i \equiv U^i{}_j\eta^j\,, \quad \tilde{\eta}_i \equiv \eta_j(U^{-1})^j{}_i$$

Still, the action is quite non-linear. Canonical quantization looks extremely difficult.

2.3 Phase space formulation and quantization

For non-linear systems, the phase space formulation is very useful **when the generator of the "dynamics" is contained in the the symmetry algebra** [2]

- ♦ Quantization can be performed at equal time **without solving the dynamical equation of motion**.
- ♦ **Dynamics can then be generated by a member of the algebra.**

$\boldsymbol{psu(2,2|4)}$ algebra contains

AdS energy operator $\boldsymbol{E = -i(P^0 - K^0)/\sqrt{2}}$

"light-cone" Hamiltonian $= \boldsymbol{-P^-}$.

[2] Emphasized and utilized in [Kazama and Yokoi] arXiv:0801.1561

□ **Apparent difficulties for the Dirac procedure**

We have the explicit action only on the gauge slice

$$\boldsymbol{\Theta_I^+ = (\theta^+, \eta^+) = 0}$$

Cannot compute the momenta $\boldsymbol{P_{\Theta_I^+}}$

Noether charges are obtained by varying the symmetry parameters (made local)

$$\Downarrow$$

♠ **Information slightly away from the gauge slice seems to be needed.**

This has been preventing systematic procedure in the past

☐ **Solution of the problem :**

♦ **We have found a way to bypass this problem**

– Compute the **Lagrange bracket** in the physical sector and then invert it to get the Dirac bracekt. This requires only the information on the gauge slice. (For details, see our paper.)

– We have developed a formula for the Noether charge which can be evaluated directly on the gauge slice.

In this way, the quantum Noether charges are constructed in terms of the **"free fields"** satisfying the canonical CR

$$[x, P_x] = [\bar{x}, P_{\bar{x}}] = \left[x^-, P_-\right] = [\phi, P_\phi] = i\,, \quad \left[y^{A'}, P_{B'}\right] = i\delta^{A'}_{B'}$$

$$\left\{S^i, S_j\right\} = \left\{\tilde{S}^i, \tilde{S}_j\right\} = \delta^i_j \qquad (S^i \leftarrow i\sqrt{P_-}\theta^i\,, \tilde{S}^i \leftarrow i\sqrt{P_-}e^\phi\eta^i)$$

3 Quantum superconformal generators

3.1 Quantum superconformal generators

Below we use the notation $N_S = S^i S_i\,, N_{\tilde S} = \tilde S^i \tilde S_i\,, S\cdot\tilde S = S^i\tilde S_i, z = e^\phi$

$$\mathbb{P}^x = iP_{\bar x}$$

$$\mathbb{P}^{\bar x} = iP_x$$

$$\mathbb{P}^+ = iP_-$$

$$\begin{aligned}\mathbb{P}^- = -H_{lc} = \frac{i}{4P_-}\Big[&-4P_xP_{\bar x} + \partial_z^2 - \frac{1}{z}\partial_z \\ &+ \frac{1}{z^2}(-3 - \hat l^2 + 4N_{\tilde S} - N_{\tilde S}^2 + 4l^m{}_k\tilde S^k\tilde S_m)\Big]\end{aligned}$$

$$\begin{aligned}\mathbb{K}^x = &-iz^2P_{\bar x} + x\left(z\partial_z + ix^-P_- + ixP_x + \frac{1}{2}(N_S - N_{\tilde S} + 3)\right) \\ &+ izS\cdot\tilde S - \tau\mathbb{J}^{-x}\end{aligned}$$

$$\begin{aligned}\mathbb{K}^{\bar x} = &-iz^2P_x + \bar x\left(z\partial_z + ix^-P_- + i\bar xP_{\bar x} + \frac{1}{2}(-N_S + N_{\tilde S} + 3)\right) \\ &+ iz\tilde S\cdot S - \tau\mathbb{J}^{-\bar x}\end{aligned}$$

$$
\begin{aligned}
\mathbb{K}^+ &= \frac{1}{i}(z^2 + x\bar{x})P_- + \tau(z\partial_z + ixP_x + i\bar{x}P_{\bar{x}} + 1 + \tau\mathbb{P}^-) \\
\mathbb{K}^- &= (x\bar{x} - z^2)\mathbb{P}^- + \bar{x}\mathbb{J}^{-x} + x\mathbb{J}^{-\bar{x}} + x^- z\partial_z + i(x^-)^2 P_- + 2x^- \\
&\quad + \frac{i}{4P_-}\Big[-2z\partial_z - 1 + 2N_{\tilde{S}} - N_{\tilde{S}}^2 - 2N_S + N_S^2 + 4(\tilde{S}\cdot S)(S\cdot\tilde{S}) \\
&\quad + 4l^k{}_m(\tilde{S}^m\tilde{S}_k - S^m S_k) - 4z(P_{\bar{x}}\tilde{S}\cdot S + P_x S\cdot\tilde{S})\Big] \\
\mathbb{D} &= -z\partial_z - (ix^- P_- + ixP_x + i\bar{x}P_{\bar{x}}) - \frac{3}{2} - \tau\mathbb{P}^- \\
\mathbb{J}^{+-} &= -ix^- P_- - \frac{1}{2} + \tau\mathbb{P}^- \\
\mathbb{J}^{x\bar{x}} &= -i\bar{x}P_{\bar{x}} + ixP_x + \frac{1}{2}(N_S - N_{\tilde{S}}) \\
\mathbb{J}^{+x} &= -ixP_- + i\tau P_{\bar{x}} \\
\mathbb{J}^{+\bar{x}} &= -i\bar{x}P_- + i\tau P_x \\
\mathbb{J}^{-x} &= -x\mathbb{P}^- + ix^- P_{\bar{x}} - \frac{P_{\bar{x}}}{2P_-}(N_S + N_{\tilde{S}} - 1) + \frac{i}{\sqrt{P_-}}S^k\mathbb{Q}_k^- \\
\mathbb{J}^{-\bar{x}} &= -\bar{x}\mathbb{P}^- + ix^- P_x + \frac{P_x}{2P_-}(N_S + N_{\tilde{S}} + 1) + \frac{i}{\sqrt{P_-}}\mathbb{Q}^{-k}S_k
\end{aligned}
$$

$$\mathbb{J}^i{}_j = l^i{}_j + M^i{}_j$$

$$l^i{}_j = \tfrac{1}{2}(U^{-1}\gamma^{A'}U)^i{}_j J^{A'}_B\,, \quad M^i{}_j \equiv S^i S_j - \tfrac{1}{4}\delta^i_j N_S + \tilde{S}^i \tilde{S}_j - \tfrac{1}{4}\delta^i_j N_{\tilde{S}}$$

$$\mathbb{Q}^{+i} = i\sqrt{P_-}S^i$$

$$\mathbb{Q}^+_i = -i\sqrt{P_-}\tilde{S}_i$$

$$\mathbb{Q}^{-i} = \frac{i}{2\sqrt{P_-}}\Big[2P_x S^i - \partial_z \tilde{S}^i + \frac{1}{z}\Big(\tilde{S}^i(N_{\tilde{S}} - 1) - 2l^i{}_k \tilde{S}^k\Big)\Big]$$

$$\mathbb{Q}^-_i = \frac{-i}{2\sqrt{P_-}}\Big[2P_{\bar{x}} S_i + \partial_z \tilde{S}_i + \frac{1}{z}\Big(\tilde{S}_i(N_{\tilde{S}} - 3) - 2\tilde{S}_k l^k{}_i\Big)\Big]$$

$$\mathbb{S}^{+i} = -i\sqrt{P_-}\Big(z\tilde{S}^i + i\bar{x}S^i\Big) + i\tau\mathbb{Q}^{-i}$$

$$\mathbb{S}^+_i = i\sqrt{P_-}\Big(z\tilde{S}_i - ixS_i\Big) - i\tau\mathbb{Q}^-_i$$

$$\mathbb{S}^{-i} = \frac{-i}{2\sqrt{P_-}}\Big[2zP_{\bar{x}}\tilde{S}^i - 2\tilde{S}^i(S\cdot\tilde{S}) - S^i(z\partial_z + N_S + 1) + 2l^i{}_k S^k\Big]$$
$$+ ix^-\mathbb{Q}^{+i} + i\bar{x}\mathbb{Q}^{-i}$$

$$\mathbb{S}^-_i = \frac{i}{2\sqrt{P_-}}\Big[2zP_x\tilde{S}^i - 2\tilde{S}^i(\tilde{S}\cdot S) + S_i(z\partial_z - N_S + 5) + 2l^k{}_i S_k\Big]$$
$$- ix^-\mathbb{Q}^+_i - i\bar{x}\mathbb{Q}^-_i$$

- These generators[3] are properly normal ordered such that
  - They satisfy the proper hermiticity properties
  - They satisfy the $\boldsymbol{psu(2,2|4)}$ algebra
- $\boldsymbol{\tau}$-dependence is generated precisely by the **unitary transformation**

$$\mathcal{O}(\tau) = e^{\tau \mathbb{P}^-} \mathcal{O} e^{-\tau \mathbb{P}^-}$$

$\Rightarrow$ We can study the system at $\boldsymbol{\tau = 0}$ first and then generate $\boldsymbol{\tau}$-dependence later by the above unitary transformation.

We wish to obtain all the physical states
forming unitary irreducible representations of $\boldsymbol{psu(2,2|4)}$
generated by this concrete system.

[3] Classically, these generators were obtained, somewhat indirectly, by Metsaev, Tseytlin and Thorn in hep-th/0009171.

4 Complete solution for superconformal primaries

4.1 Dilatation(D) scheme and energy(E) scheme for the $SO(4,2)$ sector

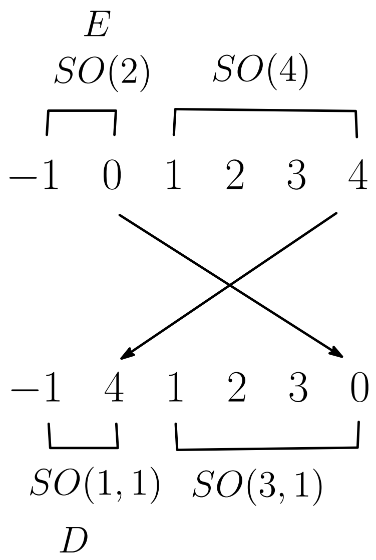
There are two schemes for the choice of the maximal subgroups of $\boldsymbol{SO(4,2)}$

$\boldsymbol{E}$-scheme $\quad \boldsymbol{SO(4,2) \supset SO(2)_E \times SU(2)_L \times SU(2)_R}$

$\boldsymbol{D}$-scheme $\quad \boldsymbol{SO(4,2) \supset SO(1,1)_D \times SL(2,C) \times \overline{SL(2,C)}}$

Transformation from the $\boldsymbol{D}$-scheme to the $\boldsymbol{E}$-scheme is effected by the **non-unitary** similarity transformation

$$\boxed{\begin{aligned} \boldsymbol{V\mathcal{O}V^{-1}} &= \boldsymbol{\hat{\mathcal{O}}} \\ \boldsymbol{V} &= \mathrm{e}^{\boldsymbol{i(\pi/2)R}} \\ \boldsymbol{R} = \boldsymbol{T^{40}} &= \text{anti-hermitian} \end{aligned}}$$



♦ This is designed to map the dilatation operator $\mathbb{D} = T^{4,-1}$ to the AdS energy operator $\mathbb{E} = -iT^{0,-1}$:

$$V\mathbb{D}V^{-1} \equiv \hat{\mathbb{D}} = \mathbb{E} = \frac{1}{2i}(\mathbb{P}^+ - \mathbb{P}^- - \mathbb{K}^+ + \mathbb{K}^-)$$

♦ As it is a similarity transformation, this mapping preserves the structure of the superconformal algebra.

However, it is a **non-unitary** transformation and hence **it does not preserve the norm**.

To compare with the supergravity result, we should look for unitary (hence normalizable) representations with real AdS energy E.

So, we label the "highest weight" state by

$$|E, j_L, j_R\rangle \quad \Leftrightarrow \quad SO(2) \times SU(2)_L \times SU(2)_R \subset SO(4,2)$$

Remark:

♦ $\mathbb{E}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle = \boldsymbol{E}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle \Rightarrow \mathbb{D}(\boldsymbol{V}^{-1}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle) = \boldsymbol{E}(\boldsymbol{V}^{-1}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle)$

$\therefore$ $\boldsymbol{V}^{-1}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle$ is an eigenstate of an anti-hermitian generator $\mathbb{D}$ with real eigenvalue.

This is consistent since

- One can show that $\boldsymbol{V}^{-1}|\boldsymbol{E}, \boldsymbol{j_L}, \boldsymbol{j_R}\rangle$ is of zero-norm.
- In the AdS/CFT context, the gauge-invariant composite operators in SYM carry real eigenvalues with respect to anti-hermitian dilatation operator.

4.2 Conformal and superconformal primary conditions

So we must formulate the conformal and superconformal primary state conditions in the $\boldsymbol{E}$-scheme.

In $\boldsymbol{D}$-scheme: Conf. primary: $\mathbb{K}|\boldsymbol{\Psi}\rangle = \boldsymbol{0}$ Superconf. primary: $\mathbb{S}|\boldsymbol{\Psi}\rangle = \boldsymbol{0}$

(Of course $\mathbb{S}|\boldsymbol{\Psi}\rangle = \boldsymbol{0} \Rightarrow \mathbb{K}|\boldsymbol{\Psi}\rangle = \boldsymbol{0}$)

In $\boldsymbol{E}$-scheme: Conf. primary: $\hat{\mathbb{K}}|\boldsymbol{\Psi}\rangle = \mathbf{0}$ Superconf. primary: $\hat{\mathbb{S}}|\boldsymbol{\Psi}\rangle = \mathbf{0}$

$$\hat{\mathbb{K}}^{x} = \frac{1}{2}(\mathbb{K}^{x} - \mathbb{P}^{x} - i(\mathbb{J}^{+x} - \mathbb{J}^{-x}))$$

$$\hat{\mathbb{K}}^{\bar{x}} = \frac{1}{2}(\mathbb{K}^{\bar{x}} - \mathbb{P}^{\bar{x}} - i(\mathbb{J}^{+\bar{x}} - \mathbb{J}^{-\bar{x}}))$$

$$\hat{\mathbb{K}}^{+} = \frac{1}{2}(\mathbb{K}^{+} - \mathbb{P}^{-} - i(\mathbb{J}^{+-} - \mathbb{D}))$$

$$\hat{\mathbb{K}}^{-} = \frac{1}{2}(\mathbb{K}^{-} - \mathbb{P}^{+} - i(\mathbb{J}^{+-} + \mathbb{D}))$$

$$\hat{\mathbb{S}}^{+i} = \frac{1}{\sqrt{2}}(\mathbb{S}^{+i} + \mathbb{Q}^{-i})\,, \qquad \hat{\mathbb{S}}^{-i} = \frac{1}{\sqrt{2}}(\mathbb{S}^{-i} - \mathbb{Q}^{+i})$$

$$\hat{\mathbb{S}}_{i}^{+} = \frac{1}{\sqrt{2}}(\mathbb{S}_{i}^{+} - \mathbb{Q}_{i}^{-})\,, \qquad \hat{\mathbb{S}}_{i}^{-} = \frac{1}{\sqrt{2}}(\mathbb{S}_{i}^{-} + \mathbb{Q}_{i}^{+})$$

$$\hat{\mathbb{Q}}^{+i} = \frac{1}{\sqrt{2}}(\mathbb{Q}^{+i} + \mathbb{S}^{-i})\,, \qquad \hat{\mathbb{Q}}^{-i} = \frac{1}{\sqrt{2}}(\mathbb{Q}^{-i} - \mathbb{S}^{+i})$$

$$\hat{\mathbb{Q}}_{i}^{+} = \frac{1}{\sqrt{2}}(\mathbb{Q}_{i}^{+} - \mathbb{S}_{i}^{-})\,, \qquad \hat{\mathbb{Q}}_{i}^{-} = \frac{1}{\sqrt{2}}(\mathbb{Q}_{i}^{-} + \mathbb{S}_{i}^{+})$$

4.3 Allowed highest weight unitary representations for the $SU(4)$ sector

We first examine what kind of highset weight unitary irrep are allowed for the $SU(4)$ sector. $SU(4)$ generators consist of the orbital and the spin part:

$$\mathbb{J}^i{}_j = l^i{}_j + M^i{}_j$$

$$l^i{}_j = \frac{1}{2}(U^{-1}\gamma^{A'}U)^i{}_j J_B^{A'}$$

$$M^i{}_j \equiv S^i S_j - \frac{1}{4}\delta^i_j N_S + \tilde{S}^i \tilde{S}_j - \frac{1}{4}\delta^i_j N_{\tilde{S}}$$

♡ **Observation: (partial) Casimir operator $\hat{l}^2 \equiv l^m{}_n l^n{}_m$ commutes with all the $psu(2,2|4)$ generators.**

⇒ **Orbital part can be analyzed independently.**

□ Orbital part:

There exists a quadratic product relation[4] satisfied by $l^i{}_j$

$$(\star) \quad \mathcal{L}^i{}_j \equiv l^i{}_k l^k{}_j - \frac{1}{4}\hat{l}^2 \delta^i_j - 2l^i{}_j = 0$$

Product relation among the generators

$$\Downarrow$$

Restriction on the highest weight module

Example: $SU(2)$ in terms of fermionic oscillators b_i, $i = 1, 2$, with $\left\{b_i^\dagger, b_j\right\} = \delta_{ij}$

$$J^3 = \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2)$$

$$J^+ = b_1^\dagger b_2 \,, \quad J^- = b_2^\dagger b_1$$

$$\Rightarrow \quad \left[J^3, J^\pm\right] = \pm J^\pm \,, \quad \left[J^+, J^-\right] = 2J^3$$

[4] This relation was found by Metsaev, hep-th/9908114

Product relation $J^- J^- = 0 \quad \Rightarrow$ HW representation is utmost 2-dimensional.

In fact we can only have 3 possible HW states:

$$|0\rangle (1 \text{ dim})\,, \qquad b_1^\dagger|0\rangle (2 \text{ dim})\,, \qquad b_1^\dagger b_2^\dagger|0\rangle (1 \text{ dim})$$

Analyze $(\star)$ in our case using the Chevalley basis generators:

$$l^i{}_j = \begin{pmatrix} \frac{1}{4}(3H_1 + 2H_2 + H_3) & E_1^+ & [E_1^+, E_2^+] & [E_1^+, [E_2^+, E_3^+]] \\ E_1^- & \frac{1}{4}(-H_1 + 2H_2 + H_3) & E_2^+ & [E_2^+, E_3^+] \\ -[E_1^-, E_2^-] & E_2^- & -\frac{1}{4}(H_1 + 2H_2 - H_3) & E_3^+ \\ [E_1^-, [E_2^-, E_3^-]] & -[E_2^-, E_3^-] & E_3^- & -\frac{1}{4}(H_1 + 2H_2 + 3H_3) \end{pmatrix}$$

where

$$\Big[H_{\hat{i}}, H_{\hat{j}}\Big] = 0\,, \qquad \Big[E_{\hat{i}}^+, E_{\hat{j}}^-\Big] = \delta_{\hat{i}\hat{j}} H_{\hat{j}}$$

$$\Big[H_{\hat{i}}, E_{\hat{j}}^\pm\Big] = \pm K_{\hat{j}\hat{i}} E_{\hat{j}}^\pm\,, \qquad \hat{i}, \hat{j} = 1, 2, 3$$

$$K_{\hat{j}\hat{i}} = \text{Cartan matrix}$$

An example of the power of the quadratic relation $(\star)$:

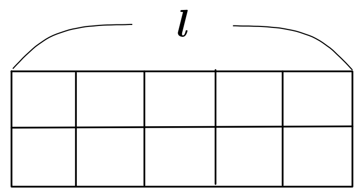
$$\boldsymbol{\mathcal{L}^2}_1|\boldsymbol{\lambda_1,\lambda_2,\lambda_3}\rangle = \mathbf{0}$$

$$\Rightarrow \quad \underbrace{(\boldsymbol{\lambda_1 + 2\lambda_2 + \lambda_3 + 2})}_{\neq 0} \boldsymbol{E_1^-}|\boldsymbol{\lambda_1,\lambda_2,\lambda_3}\rangle = \mathbf{0}$$

$$\Rightarrow \quad \boldsymbol{\lambda_1 = 0}$$

Result: The only allowed HWS are

$$\boxed{\begin{array}{l} |\mathbf{0},\boldsymbol{l},\mathbf{0}\rangle\,, \qquad \boldsymbol{l} = \mathbf{0,1,2,\ldots} \\ \qquad\qquad \hat{\boldsymbol{l}}^2|\mathbf{0},\boldsymbol{l},\mathbf{0}\rangle = \boldsymbol{l(l+4)}|\mathbf{0},\boldsymbol{l},\mathbf{0}\rangle \end{array}}$$



□ Inclusion of the spin part:

Spin part of the Hilbert space is generated by $\boldsymbol{S^i}$ and $\boldsymbol{\tilde{S}^i}$ upon the vacuum $|\mathbf{0}\rangle$.

$\Rightarrow$ $\mathbf{2^8}$ states: decomposed into many different HWR's

$\Leftrightarrow$ Full generators $\mathbb{J}^{\boldsymbol{i}}{}_{\boldsymbol{j}}$ do not satisfy a simple product relation like $(\star)$.

But, if we combine with some of the superconformal primary conditions, we can derive

$$\mathcal{J}^i_j \equiv \mathbb{J}^i{}_k \mathbb{J}^k{}_j - \frac{1}{4}\hat{\mathbb{J}}^2 \delta^i_j - \left(4 - \frac{N}{2}\right)\mathbb{J}^i{}_j \approx 0$$

$$N \equiv N_S + N_{\tilde{S}} \qquad (N = 0, 1, \ldots, 8)$$

where $\quad \hat{\mathbb{J}}^2 \equiv \mathbb{J}^i{}_k \mathbb{J}^k{}_i \approx 4\mathbb{E} + \hat{l}^2 - \frac{1}{4}(N-4)^2 + 4$

- Repeat the analysis as before for each $N \Rightarrow$ Many representations eliminated.
- Utilize unitarity bounds which follow from

$$\langle \Psi | \left\{ \hat{\mathbb{S}}^-_i , \hat{\mathbb{Q}}^{+i} \right\} | \Psi \rangle = |\hat{\mathbb{Q}}^{+i} |\Psi\rangle|^2 \geq 0 \,, \qquad etc$$

Example: $\quad E \geq \lambda_1 + \lambda_2 + \lambda_3$, etc.

Result: Allowed HW states are

$$(i) \quad |\Omega_l\rangle = S^1 \tilde{S}^1 S^2 \tilde{S}^2 |0\rangle \otimes |0, l, 0\rangle \quad \sim [0, l+2, 0]$$

$$(ii) \quad |\mathrm{vac}\rangle = |0\rangle \otimes |0, 0, 0\rangle$$

$$(iii) \quad |\mathrm{fvac}\rangle = S^1 \tilde{S}^1 S^2 \tilde{S}^2 S^3 \tilde{S}^3 S^4 \tilde{S}^4 |0\rangle \otimes |0, 0, 0\rangle$$

(ii) and (iii) eventually lead to non-normalizable states.

4.4 Solution of the superconformal primaries

4.4.1 Solution of the superconformal primaries at $\tau = 0$

On $|\Omega_l\rangle$, the supercharge operators effectively simplify substantially:

Below we use the indices $i = (\alpha, \hat{\alpha})$, $\alpha = 1, 2$, $\hat{\alpha} = 3, 4$.

$$
\begin{aligned}
\mathbb{Q}^{\pm\alpha} &= 0 \\
\mathbb{Q}^{\pm}_{\hat{\alpha}} &= 0 \\
\mathbb{Q}^{+\hat{\alpha}} &= i\sqrt{P_-}S^{\hat{\alpha}} \\
\mathbb{Q}^{+}_{\alpha} &= -i\sqrt{P_-}\tilde{S}_{\alpha} \\
\mathbb{Q}^{-\hat{\alpha}} &= \frac{i}{2\sqrt{P_-}}\left(2P_x S^{\hat{\alpha}} - \left(\partial_z - \frac{l+1}{z}\right)\tilde{S}^{\hat{\alpha}}\right) \\
\mathbb{Q}^{-}_{\alpha} &= \frac{-i}{2\sqrt{P_-}}\left(2P_{\bar{x}} S_{\alpha} + \left(\partial_z - \frac{l+1}{z}\right)\tilde{S}_{\alpha}\right)
\end{aligned}
$$

$$\mathbb{S}^{\pm\alpha} = \mathbf{0}$$

$$\mathbb{S}^{\pm}_{\hat{\alpha}} = \mathbf{0}$$

$$\mathbb{S}^{+\hat{\alpha}} = -i\sqrt{P_-}\left(z\tilde{S}^{\hat{\alpha}} - \frac{\partial}{\partial P_{\bar{x}}}S^{\hat{\alpha}}\right)$$

$$\mathbb{S}^{+}_{\alpha} = i\sqrt{P_-}\left(z\tilde{S}_{\alpha} + \frac{\partial}{\partial P_{x}}S_{\alpha}\right)$$

$$\mathbb{S}^{-\hat{\alpha}} = \frac{-i}{2\sqrt{P_-}}\left[2zP_{\bar{x}}\tilde{S}^{\hat{\alpha}} - (z\partial_z + l + 3)S^{\hat{\alpha}}\right] - \frac{\partial}{\partial P_-}\mathbb{Q}^{+\hat{\alpha}} - \frac{\partial}{\partial P_x}\mathbb{Q}^{-\hat{\alpha}}$$

$$\mathbb{S}^{-}_{\alpha} = \frac{i}{2\sqrt{P_-}}\left[2zP_{x}\tilde{S}_{\alpha} + (z\partial_z + l + 3)S_{\alpha}\right] + \frac{\partial}{\partial P_-}\mathbb{Q}^{+}_{\alpha} + \frac{\partial}{\partial P_{\bar{x}}}\mathbb{Q}^{-}_{\alpha}$$

We will seek the superconformal primary state $|\Psi\rangle$ in the form

$$|\Psi\rangle = \Phi(z, P_-, P_x, P_{\bar{x}})|\Omega_l\rangle$$

- The following half of the superconformal conditions are automatic:

$$\hat{\mathbb{S}}^{\pm\alpha}|\Psi\rangle = 0\,, \qquad \hat{\mathbb{S}}^{\pm}_{\hat{\alpha}}|\Psi\rangle = 0$$

- Similarly, the following 8 conditions hold automatically:

$$\hat{\mathbb{Q}}^{\pm\alpha}|\Psi\rangle = 0\,, \qquad \hat{\mathbb{Q}}^{\pm}_{\hat{\alpha}}|\Psi\rangle = 0$$

This shows that $|\Psi\rangle$ **must be half BPS**.

We now impose the remaining superconformal primary conditions one by one to determine the form of Φ .

(1) First, consider

$$\begin{aligned} 0 &= \sqrt{2}\,\hat{\mathbb{S}}^{+\hat{\alpha}}|\Psi\rangle = (\mathbb{S}^{+\hat{\alpha}} + \mathbb{Q}^{-\hat{\alpha}})|\Psi\rangle \\ &= \frac{i}{2\sqrt{P_-}}\left[2\left(P_x + P_-\frac{\partial}{\partial P_{\bar{x}}}\right)S^{\hat{\alpha}} - \left(\partial_z - \frac{l+1}{z} + 2P_- z\right)\tilde{S}^{\hat{\alpha}}\right]|\Psi\rangle \end{aligned}$$

From the coefficient of $S^{\hat{\alpha}}$ and $\tilde{S}^{\hat{\alpha}}$, we get two simple differential equations

$$(i) \qquad \left(P_x + P_- \frac{\partial}{\partial P_{\bar{x}}}\right)\Phi = 0$$

$$(ii) \qquad \left(\partial_z - \frac{l+1}{z} + 2P_- z\right)\Phi = 0$$

They determine the dependence on $P_{\bar{x}}$ and z as

$$\Phi = f(P_-)\psi$$

$$(\star 1) \quad \psi = \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} - z^2 P_-\right) z^{l+1}$$

(2) Next consider the following condition

$$0 = \sqrt{2}\,\hat{\mathbb{S}}^{-\hat{\alpha}}|\Psi\rangle = (\mathbb{S}^{-\hat{\alpha}} - \mathbb{Q}^{+\hat{\alpha}})|\Psi\rangle$$

This gives a simple differential equation with respect to P_-:

$$\begin{aligned}&(\mathbb{S}^{-\hat{\alpha}} - \mathbb{Q}^{+\hat{\alpha}})|\Psi\rangle \\ &= -\frac{i}{\sqrt{P_-}}\left[(1+z^2)P_- - \left(l + \frac{1}{2}\right) - \frac{P_x P_{\bar{x}}}{P_-} + P_- \frac{\partial}{\partial P_-}\right] S^{\hat{\alpha}}|\Psi\rangle = 0\end{aligned}$$

This determines $f(P_-)$ as

$$f(P_-) = ce^{-P_-} P_-^{l+(1/2)} \,, \quad c = \text{constant}$$

(3) The remaining conditions $0 = \sqrt{2}\,\hat{\mathbb{S}}_\alpha^+|\Psi\rangle = (\mathbb{S}_\alpha^+ - \mathbb{Q}_\alpha^-)|\Psi\rangle$ and $0 = \sqrt{2}\,\hat{\mathbb{S}}_\alpha^- \Psi = (\mathbb{S}_\alpha^- + \mathbb{Q}_\alpha^+)|\Psi\rangle$ are satisfied automatically.

□ Complete solution at $\tau = 0$:

Thus we found all the unitary superconformal primary states in the form[5]

$$\boxed{\begin{aligned} |\Psi_l\rangle = C_l \exp\left(-\frac{P_x P_{\bar{x}}}{P_-} - (z^2+1)P_-\right) z^{l+1} P_-^{l+(1/2)} \\ \times S^1 \tilde{S}^1 S^2 \tilde{S}^2 |0\rangle |0, l, 0\rangle \,, \qquad l = 0, 1, 2, \ldots \end{aligned}}$$

[5] For some special states belonging to $l = 0$ multiplet, Metsaev obtained the bosonic part of the wave function in hep-th/0201226.

Properties of $|\Psi_l\rangle$

- Quantum numbers of $|\Psi_l\rangle$:

$$\text{AdS energy} \quad \mathbb{E}|\Psi_l\rangle = E_l|\Psi_l\rangle\,, \qquad E_l = l+2$$

$$SU(2)_L \times SU(2)_R \text{ spins} \quad \mathbb{J}^3_{L,R}|\Psi_l\rangle = 0$$

$$SU(4)\ \text{Casimir} \quad \hat{\mathbb{J}}^2|\Psi_l\rangle = (l+2)(l+6)|\Psi_l\rangle$$

- AdS "mass" formula : m^2 operator $\equiv$ quadratic Casimir of $SO(4,2)$

$$\frac{1}{2}T^{AB}T_{AB}|\Psi_l\rangle = E_l(E_l-4)|\Psi_l\rangle$$

- Measure and normalizability: Quantum mechanical measure must be such as to respect the hermiticity of the basic variables. The range of P_- should be taken as $[0,\infty]$, as $|\Psi_l\rangle$ vanishes at both end points.

$$\int_0^\infty \underbrace{\frac{dz}{z}}_{d\phi} \int_0^\infty dP_- \int_{-\infty}^\infty dP_1 \int_{-\infty}^\infty dP_2\, \langle\Psi_l|\Psi_l\rangle = |C_l|^2 \frac{(l+1)(l!)^2}{2^{2l+4}}\pi$$

Representations built on $|\Psi_l\rangle$

- The states of the representation are produced by operating the 8 supercharges $\hat{\mathbb{Q}}^{\pm 3,4}, \hat{\mathbb{Q}}^{\pm}_{1,2}$ (and the momentum operators $\hat{\mathbb{P}}$'s), with the property $\left[\mathbb{E}, \hat{\mathbb{Q}}\right] = \frac{1}{2}\hat{\mathbb{Q}}$.

Example: $\hat{\mathbb{Q}}^{+3}|\Psi_l\rangle = i\sqrt{2P_-}S^3|\Psi_l\rangle$.

- Dimension of the representation (up to the action of $\hat{\mathbb{P}}$'s)

$$2^8 \times \dim[0, l, 0] = \frac{64}{3}(l+1)(l+2)^2(l+3)$$

These are precisely the $\frac{1}{2}$ BPS superconformal multiplets of 1-particle states realized in supregravity

$\Longleftrightarrow$ Single trace operators Tr $(\phi^{\{I_1}\phi^{I_2}\cdots\phi^{I_{l+2}\}})$ and its descendants in SYM.

□ Complete solution at arbitrary τ :

The solution at arbitrary τ is obtained by the unitary transformation

$$|\Psi_l(\tau)\rangle = e^{\tau \mathbb{P}^-}|\Psi_l\rangle$$

$$\mathbb{P}^- = \frac{i}{4P_-}\left(D_z^{(l)} - 4P_x P_{\bar{x}}\right)$$

$$D_z^{(l)} \equiv \partial_z^2 - \frac{1}{z}\partial_z - \frac{l^2-1}{z^2} \qquad \text{(Bessel-type operator)}$$

$\Leftrightarrow$ $|\Psi_l(\tau)\rangle$ satisfies the Schrödinger equation

$$\partial_\tau |\Psi_l(\tau)\rangle = \mathbb{P}^-|\Psi_l(\tau)\rangle$$

$$\Leftrightarrow \qquad (*) \quad 4\left(\frac{1}{i}P_-\partial_\tau + P_x P_{\bar{x}}\right)|\Psi_l(\tau)\rangle = D_z^{(l)}|\Psi_l(\tau)\rangle$$

$\Downarrow$ (“Bessel transform”)

Completely explicit wave function of the superconformal primary states for a superparticle in $AdS_5 \times S^5$

$$
\begin{aligned}
|\Psi_l(\tau)\rangle = C_l \left(\frac{z}{1+i\tau}\right)^{l+1} P_-^{l+(1/2)} \\
\times \exp\left(-\frac{P_x P_{\bar{x}}}{P_-}(1+i\tau) - P_- - \frac{z^2 P_-}{1+i\tau}\right) \\
\times S^1 \tilde{S}^1 S^2 \tilde{S}^2 |0\rangle \otimes |0, l, 0\rangle
\end{aligned}
$$

5 On-going and future Projects

- **Extension to the string case**
  - Quantum superconformal algebra
  - Solutions of superconformal primary conditions
  - Expect to meet some integrable structure in diagonalizing the spectrum
- **Understanding of GKP-W relation from 1st quantized point of view**
  - Constructtion of vertex operators $\mathcal{V}_I(x)$ anchored at a point on the boundary.

$$\langle \mathcal{V}_{I_1}(x_1) \cdots \mathcal{V}_{I_n}(x_n) \rangle_{particle} \sim \langle \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n) \rangle_{CFT} \ ?$$

- Quantum superparticle in other spaces such as $AdS_4 \times S^7$, etc.

We wish to report progress on these and related matters in the near future

and

Thank you all for making the Komaba 2010 workshop a fruitful one !