Large N reduction on group manifolds and coset spaces

Hikaru Kawai, Asato Tsuchiya and Shinji Shimasaki.
(Kyoto Univ.) (Shizuoka Univ.) (Harish Chandra Inst.)

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What I would like to show:

“Large-N reduction works for group manifolds in its original form.”

Matrix model

\[ S = -\frac{\nu}{4\kappa^2} Tr \left( \left[ \hat{X}_a, \hat{X}_b \right] - i f_{abc} \hat{X}_c \right)^2, \]

\[ f_{abc} : \text{structure constant of a group } G \]

is equivalent to YM theory on G, if we expand \( \hat{X}_a \) around a special minimum of S, and take the large-N limit.

Minimum of S: \[ \left[ \hat{X}_a, \hat{X}_b \right] = i f_{abc} \hat{X}_c \Rightarrow \hat{X}_a \text{ are rep. matrices of } G. \]

We consider a special representation of the form \( \hat{L}_a = T_a^{(\text{reg})} \otimes 1_k \),

where \( T_a^{(\text{reg})} \) is the rep. matrix for the regular representation of G, and \( 1_k \) is k x k unit matrix.

\( T_a^{(\text{reg})} \) is first truncated to n dims, and take the limit \( n, k \to \infty. \)

NB Different from fuzzy manifolds.

D-dimensional space-time emerges from D matrices.
Introduction

- Large N reduction
  - large-N gauge theory is equivalent to lower dimensional models obtained by dimensional reduction.

- Conceptual importance: Emergence of space-time from matrix degrees of freedom.
- Practical use: Non-perturbative formulation of large N gauge theory.
  - In particular, super symmetric Yang Mills theory.

- So far it has been investigated mainly on flat space-time.

- Generalization to $S^3$ was done. Ishii-Ishiki-Shimasaki-Tsuchiya. (’08)
  - They have considered N=4 SYM on $R \times S^3$.

- General curved spacetime?
  - Description of curved space-times by matrices. Hanada-Kawai-Kimura(’06)
  - Fluctuation around them is still not clear.

- Here we show that the large N reduction works on group manifolds and coset spaces.
  - Usually large N reduction is shown in momentum space.
  - We reconsider it in coordinate space to make the generalization easier.
  - We consider a kind of bi-local field theory.
Outline

1. Introduction
2. Bi-local field theory interpretation of reduced model
3. Large N reduction on group manifolds
4. Large N reduction for N=4 SYM on $\mathbb{R} \times S^3$
5. Summary and outlook
Bi-local field theory interpretation of reduced model
phi^3 matrix field theory on R^d

**Action**

\[ S = \int d^d x \ Tr \left( \frac{1}{2} (\partial_\mu \phi(x))^2 + \frac{1}{2} m^2 \phi(x)^2 + \frac{1}{3} \kappa \phi(x)^3 \right) \]

\[ \phi(x) : N \times N \text{ hermitian matrix} \]

**Propagator**

\[ \langle \phi(x_1)_{ij} \phi(x_2)_{kl} \rangle = D(x_1 - x_2) \delta_{il} \delta_{jk} \]

**Vertex**

\[ -\kappa \ \delta_{ij} \delta_{kl} \delta_{mn} \]

**Large N limit**

\[ N \to \infty, \ \kappa \to 0 \text{ with } \kappa^2 N = \lambda \text{ fixed} \]
phi^3 matrix field on R^d (cont’d)

Free energy at the two-loop level

Planar

\[ x_1 \quad \quad \quad \quad \quad x_2 = \frac{1}{6} N^2 \lambda \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3 \]

Non-planar

\[ x_1 \quad \quad \quad \quad \quad x_2 = \frac{1}{6} \lambda \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3 \]

Suppressed
Large N reduction

\[ S = \int d^d x \ \text{Tr} \left( \frac{1}{2} (\partial_\mu \phi(x))^2 + \frac{1}{2} m^2 \phi(x)^2 + \frac{1}{3} \kappa \phi(x)^3 \right) \]

**reduced model**

Construct a matrix model by the following procedure:

\[ \phi(x) \rightarrow \tilde{\phi}, \quad \partial_\mu \rightarrow [i \tilde{P}_\mu, \ ] , \quad \int d^d x \rightarrow v \]

\[ S_r = v \text{Tr} \left( \frac{1}{2} [i \tilde{P}_\mu, \tilde{\phi}]^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{3} \kappa \tilde{\phi}^3 \right) \]

More concretely,

\( \tilde{\phi} \) : hermitian operator acting on the function space on \( \mathbb{R}^d \)

\[ \tilde{P}_\mu |x\rangle = -\frac{1}{i} \frac{\partial}{\partial x^\mu} |x\rangle , \quad \langle x| \tilde{P}_\mu = \frac{1}{i} \frac{\partial}{\partial x^\mu} \langle x| \]

\( |x\rangle \ (x \in \mathbb{R}^d) \) : coordinate basis.
Large N reduction (cont’d)

IR and UV cut off

Assume the volume of space-time is \( V \).

Function space on space-time \( \to \) N dimensional vector space.

Space-time is divided into N cells of volume \( v = \frac{V}{N} \).

Cut off momentum is given by \( v = \left( \frac{2\pi}{\Lambda} \right)^d \).

Ordinary proof

Momentum representation.

\[
\hat{P}_\mu = \begin{pmatrix} p^{(1)}_\mu \\ p^{(2)}_\mu \\ \vdots \\ p^{(N)}_\mu \end{pmatrix}
\]

\( \hat{\phi} : N \times N \) hermitian matrix.

\((p_1^{(i)}, p_2^{(i)})\)

Uniform distribution
Reduced model as a bi-local field theory

Bi-local field theory

\[ S_r = v \text{Tr} \left( \frac{1}{2} [i \tilde{P}_\mu, \tilde{\phi}]^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} \kappa \phi^3 \right) \]

\[ \langle x | \phi | x' \rangle \equiv \phi(x, x') \quad \text{Bi-local field.} \]

\[ \tilde{P}_\mu |x\rangle = -\frac{1}{i} \frac{\partial}{\partial x^\mu} |x\rangle, \quad \langle x | \tilde{P}_\mu = \frac{1}{i} \frac{\partial}{\partial x^\mu} \langle x | \]

\[ S_r = v \int d^d x d^d x' \left( -\frac{1}{2} \phi(x', x) \left( \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu} \right)^2 \phi(x, x') + \frac{1}{2} m^2 \phi(x', x) \phi(x, x') \right) \]

\[ + v \int d^d x d^d x' d^d x'' \frac{1}{3} \kappa_T \phi(x, x') \phi(x', x'') \phi(x'', x) \]

Coordinate representation.

Change of variables

\[ X^\mu = x^\mu, \quad \xi^\mu = x^\mu - x'^\mu \]

\[ \left( \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu} \right) \phi(x, x') = \frac{\partial}{\partial X^\mu} \phi(x, x') \]
Perturbative expansion in coordinate space

Propagator

$$\langle \phi(x_1, x'_1) \phi(x'_2, x_2) \rangle = \frac{1}{v} D(x_1 - x_2) \delta^d((x_1 - x'_1) - (x_2 - x'_2))$$

End points propagate like particles.
Relative coordinate \(x_1 - x'_1\) conserves.

\[x_1 - x_2 = x'_1 - x'_2\]

End points are transported parallel.

Vertex
Free energy at the two-loop level

\[ \frac{1}{6} N^2 \lambda \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3 \]

\[ \frac{1}{N^2 V} \]

\[ \frac{\kappa^2}{6 v} \delta^d(0) V^2 \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3 \]

\[ V = N v \quad \delta^d(0) = \frac{1}{v} \]

\[ \times \frac{1}{N^2 v} \]
Non-planar

Free energy at the two-loop level (cont’d)

\[
\frac{\kappa^2}{6v} \delta^d(0) \int d^d x_1 d^d x_1' d^d x_2 d^d x_2' D(x_1 - x_2) D(x_1' - x_2') D(x_1 - x_2') \\
\times \frac{1}{N^2 v} \times \frac{1}{N^2 V} \leq \frac{1}{6} \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3
\]

Suppressed by factor \(1/V^2\) compared with planar diagrams.

\[
\frac{\kappa^2}{6v} \delta^d(0) V^2 \int d^d x_1 d^d x_2 \ D(x_1 - x_2)^3
\]
Correspondence between reduced model and original theory

Limit in reduced model

\[ N \rightarrow \infty, \quad \kappa \rightarrow 0, \quad v \rightarrow 0 \quad \text{with} \quad V = Nv \rightarrow \infty, \quad \lambda = \kappa^2 N \quad \text{fixed} \]

Reduced model reproduces original field theory.

\[
\begin{align*}
\frac{F}{N^2 V} &= \frac{F_r}{N^2 v} \\
\frac{1}{N^{q/2+1}} \langle \text{Tr}(\phi(x_1)\phi(x_2)\cdots \phi(x_q)) \rangle &= \frac{1}{N^{q/2+1}} \langle \text{Tr}(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots \hat{\phi}(x_q)) \rangle_r \\
\hat{\phi}(x) &= e^{iP_{\mu}x^\mu} \phi e^{-iP_{\nu}x^\nu}
\end{align*}
\]
Large N reduction on Torus $T^d$

Torus has finite volume $V$.  No $1/V$ suppression.

Introduce *another* index in order to suppress non-planar diagrams.

\[
\hat{\phi} \quad \text{Matrix valued bi-local operator:} \\
\phi(x, x') \to \phi(x, x')_{\alpha\beta} \quad (\alpha, \beta = 1, \cdots, k) \\
\hat{P}_\mu \to \hat{P}_\mu \otimes 1_k
\]

Function space on the torus  $n$-dim vector space.
Dimension of $\hat{\phi}$ is $N=nk$.

Large-N limit of the reduced model:

\[
n \to \infty, \quad k \to \infty, \quad \kappa \to 0, \quad \text{with} \quad \lambda = \kappa^2 N = \kappa^2 nk \quad \text{fixed}
\]

Non-planar diagrams are suppressed by $1/k^2$, and the reduced model reproduces the original field theory in the planar limit.
Large N reduction for gauge theory

Apply the rule to the field strength

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \rightarrow i[\vec{P}_\mu + \vec{A}_\mu, \vec{P}_\nu + \vec{A}_\nu] = i[\vec{X}_\mu, \vec{X}_\nu] \]

\[ \partial_\mu \rightarrow [i\vec{P}_\mu, \quad ] \]

\[ A_\mu \rightarrow \vec{A}_\mu \]

\[ \vec{X}_\mu = \vec{P}_\mu + \vec{A}_\mu \]

Reduced model of YM theory

\[ S'_r = -\frac{v}{4\kappa^2} \text{Tr}[\vec{X}_\mu, \vec{X}_\nu]^2 \]

Dimensional reduction of YM theory to zero dimension.

\[ \vec{P}_\mu \text{ is interpreted as a background of } \vec{X}_\mu . \]

This background is unstable because of massless modes.

Quenching, Twisting,...

Not consistent with SUSY.

Bhanot-Heller-Neuberger ('82)  Gross-Kitazawa ('82)
Large $N$ reduction on group manifolds
Notes on group manifolds

**Lie group**

G: compact and connected Lie group. (Later we will assume G is semi-simple.)

\[ t_a \ (a = 1, \cdots, \dim G) : \text{Generators of } G. \quad [t_a, t_b] = i f_{ab}^c t_c \]

\[ |g\rangle \ (g \in G) : \text{Coordinate basis of the function space on } G. \]

**Left and right translations**

Left translation: \[ \tilde{U}_L(h)|g\rangle = |hg\rangle, \quad \langle g|\tilde{U}_L(h) = \langle h^{-1}g| \]

Right translation: \[ \tilde{U}_R(h)|g\rangle = |gh^{-1}\rangle, \quad \langle g|\tilde{U}_R(h) = \langle gh| \]

For a function on G \[ \psi(g) = \langle g|\psi\rangle, \]

\[ (\tilde{U}_L(h)\psi)(g) = \psi(h^{-1}g), \quad (\tilde{U}_R(h)\psi)(g) = \psi(gh) \]
Notes on group manifolds (cont’d)

Killing vectors

Right invariant Killing vector $\hat{L}_a$: $e^{i\epsilon \hat{L}_a} = \hat{U}_L(e^{i\epsilon a})$ infinitesimal left translation

Left

$\hat{K}_a$: $e^{i\epsilon \hat{K}_a} = \hat{U}_R(e^{i\epsilon a})$

Comm. Rel. $[\hat{L}_a, \hat{L}_b] = if_{ab}^c \hat{L}_c$, $[\hat{K}_a, \hat{K}_b] = if_{ab}^c \hat{K}_c$, $[\hat{L}_a, \hat{K}_b] = 0$

In terms of differential operators,

$\hat{L}_a |g\rangle = -\mathcal{L}_a |g\rangle$, $\langle g|\hat{L}_a = \mathcal{L}_a \langle g|$

$\hat{K}_a |g\rangle = -\mathcal{K}_a |g\rangle$, $\langle g|\hat{K}_a = \mathcal{K}_a \langle g|$

Group version of $\hat{P}_\mu |x\rangle = -\frac{1}{i} \frac{\partial}{\partial x^\mu} |x\rangle$, $\langle x|\hat{P}_\mu = \frac{1}{i} \frac{\partial}{\partial x^\mu} \langle x|$. 
Invariant 1-forms

\[
\begin{align*}
\text{Right inv. 1-form} & \quad e^a_s \\
\text{Left} & \quad s^a
\end{align*}
\]

Maurer-Cartan equation

\[
d e^a = \frac{1}{2} f_{bc}^a e^b \wedge e^c = 0, \quad d s^a = \frac{1}{2} f_{bc}^a s^b \wedge s^c = 0
\]

Right and left invariant metric

\[
h_{\mu\nu} = e^a_{\mu} e^a_{\nu} = s^a_{\mu} s^a_{\nu}
\]

Haar measure

\[
dg = d^{\text{dim } G} x \sqrt{h} = e^1 \wedge e^2 \wedge \cdots \wedge e^{\text{dim } G}
\]

Left right invariant.

volume \quad V = \int dg
phi$^3$ matrix scalar field theory on G

Scalar phi$^3$ theory on G

\[ h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = - (L_\alpha \phi)^2 \]

\[ S = \int dg \, \text{Tr} \left( -\frac{1}{2} (L_\alpha \phi(g))^2 + \frac{1}{2} m^2 \phi(g)^2 + \frac{1}{3} \kappa \phi(g)^3 \right) \]

\( \phi(g) : N \times N \) hermitian, each element is a function on G.

Possesses G x G symmetry.

Propagator

\[ \langle \phi(g_1)_{ij} \phi(g_2)_{kl} \rangle = \Delta(g_1 g_2^{-1}) \delta_{il} \delta_{jk} \]

Right G invariance

Vertex

\[ -\kappa \delta_{ij} \delta_{kl} \delta_{mn} \]
Large N reduction on $G$

$$S = \int dg \ Tr\left( -\frac{1}{2}(L_a \phi(g))^2 + \frac{1}{2}m^2 \phi(g)^2 + \frac{1}{3} \kappa \phi(g)^3 \right)$$

**Reduced model**

Construct a matrix model by the following procedure:

$$\phi(g) \rightarrow \tilde{\phi}, \quad L_a \rightarrow [\tilde{L}_a \otimes 1_k, ], \quad \int dg \rightarrow v$$

Consider the tensor product of the function space on $G$ and a k-dim vector space,

$\tilde{\phi}$ : hermitian operator acting on the tensor space.

$$S_r = v \ Tr\left( -\frac{1}{2}[\tilde{L}_a, \tilde{\phi}]^2 + \frac{1}{2}m^2 \tilde{\phi}^2 + \frac{1}{3} \kappa \tilde{\phi}^3 \right)$$
Reduced model as a bi-local field theory

Bi-local field theory
Coordinate representation.

\[ S_r = v \text{Tr} \left( -\frac{1}{2} [\hat{L}_a, \phi]^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} \kappa \phi^3 \right) \]

\[
\langle g|\tilde{\phi}|g'\rangle \equiv \phi(g, g') : \text{bi-local kxk matrix field on } G
\]

\[
\tilde{L}_a|g\rangle = -\mathcal{L}_a|g\rangle, \quad \langle g|\tilde{L}_a = \mathcal{L}_a\langle g|
\]

\[
S_r = v \int dg dg' \text{ tr} \left\{ \frac{1}{2} \phi(g', g) \left( \mathcal{L}_a^{(g)} + \mathcal{L}_a^{(g')} \right)^2 \phi(g, g') + \frac{1}{2} m^2 \phi(g', g) \phi(g, g') \right\}
\]

\[
+ v \int dg dg' dg'' \frac{1}{3} \kappa \text{ tr} (\phi(g, g') \phi(g', g'') \phi(g'', g))
\]

Change of variables

\[ u = g, \quad \zeta = g'^{-1}g \quad \rightarrow \quad (\mathcal{L}_a^{(g)} + \mathcal{L}_a^{(g')}) \phi(g, g') = \mathcal{L}_a^{(u)} \phi(g, g') \]

Haar measure is invariant.
Perturbative expansion

Propagator

\[ \langle \phi(g_1, g'_1)_{\alpha\beta} \phi(g'_2, g_2)_{\gamma\delta} \rangle = \frac{1}{v} \Delta(g_1^{-1}g_2) \delta(g'_1^{-1}g_1, g'_2^{-1}g_2) \delta_{\alpha\delta} \delta_{\beta\gamma} \]

End points propagate as particles on G.
The relative coordinate conserves during propagation.
The situation is the same as in the flat space, and the same analysis holds.

Large N reduction holds on G.

All we need is the right G invariance.
⇔ Action is written in terms of left derivatives.
UV regularization

The function space on $G$ is identified with the representation space of the regular representation.

$$(\hat{U}_L(h)\psi)(g) = \psi(h^{-1}g), \quad (\hat{U}_R(h)\psi)(g) = \psi(gh)$$

Peter-Weyl’s theorem

$$\psi(g) = \sum_r \sum_{ij} c_{ij}^{[r]} R_{ij}^{[r]}(g) \quad \text{r runs for all irreducible representations.}$$

$$V_{reg} = \bigoplus_r V_r \otimes V_{r^*}$$

$$\tilde{L}_a = \bigoplus_r L_{\tilde{a} r} \otimes 1_{d_r}, \quad \tilde{K}_a = \bigoplus_r 1_{d_r} \otimes L_{\tilde{a} r^*}$$

$L_{\tilde{a} r}$ : rep. matrix of $t_a$ in the rep. $r$

$d_r$ : dimension of the rep. $r$

Corresponding to UV cut off $\Lambda$, introduce $I_\Lambda = \{r; C_2(r) < \Lambda^2\}$.

Restrict the sum to $I_\Lambda$. Preserves $G \times G$ symmetry.
Correspondence between reduced model and original theory

\[ S_r = v \text{Tr} \left( -\frac{1}{2} [\hat{L}_a, \hat{\phi}]^2 + \frac{1}{2} m^2 \hat{\phi}^2 + \frac{1}{3} \kappa \hat{\phi}^3 \right) \]

\[ \hat{\phi} : \text{NxN hermitian matrix} \quad \hat{L}_a = \left( \bigoplus_{r \in I} L_a^{[r]} \otimes 1_{d_r} \right) \otimes 1_k \]

\[ \begin{cases} n = \sum_{r \in I} d_r^2 & \text{Function space on } G \sim n\text{-dim vector space} \\ N = nk & \text{Size of the matrix} \\ v = V/n & \text{Volume of each cell} \end{cases} \]

Limit: \( \Lambda \to \infty (n \to \infty), \quad k \to \infty, \quad \kappa \to 0, \quad \text{with } \lambda = \kappa^2 N = \kappa^2 nk \text{ fixed} \)

The reduced model reproduces the original theory in the planar limit.

Correlation functions

\[ \frac{1}{Nq/2+1} \langle \text{Tr}(\phi(x_1)\phi(x_2)\cdots\phi(x_q)) \rangle = \frac{1}{Nq/2+1} \langle \text{Tr}(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_q)) \rangle_r \]

\[ \hat{\phi}(g) = e^{i\theta^a L_a} \hat{\phi} e^{i\theta^b L_b} \text{ for } g = e^{i\theta^a t_a} \]
Example: $G = SU(2) = S^3$

\[
L_a = \begin{pmatrix}
L_a^{[0]} \\
L_a^{[1/2]} \otimes 1_2 \\
\vdots \\
L_a^{[K]} \otimes 1_{2^K + 1}
\end{pmatrix} \otimes 1_k
\]

\[
n = \sum_{j=0}^{K} (2j + 1)^2
\]

\[
v = 16\pi^2 / n
\]

\[
N = nk
\]

Preserves $SO(4) = SU(2) \times SU(2)$ symmetry.
Gauge theories on group manifolds

Expand gauge fields by the right invariant 1-form, and use Maurer-Cartan equation.

\[ A = X_a e^a \]

\[ F = dA + iA \wedge A \]

\[ = \frac{1}{2} (i\mathcal{L}_a X_b - i\mathcal{L}_b X_a + f_{ab}^c X_c + i[X_a, X_b]) e^a \wedge e^b \]

YM action

\[ S = \frac{1}{4\kappa^2} \int \text{Tr}(F \wedge *F) \]

\[ = -\frac{1}{4\kappa^2} \int dg \text{ Tr}(\mathcal{L}_a X_b - \mathcal{L}_b X_a - i f_{ab}^c X_c + [X_a, X_b])^2 \]

Reduced model

\[ S_r = -\frac{v}{4\kappa^2} \text{ Tr}([\tilde{L}_a, \tilde{X}_b] - [\tilde{L}_b, \tilde{X}_a] - i f_{ab}^c \tilde{X}_c + [\tilde{X}_a, \tilde{X}_b])^2 \]

\[ = -\frac{v}{4\kappa^2} \text{ Tr}([\tilde{L}_a + \tilde{X}_a, \tilde{L}_b + \tilde{X}_b] - i f_{ab}^c (\tilde{L}_c + \tilde{X}_c))^2 \]

Absorb the background L to X.

\[ S'_r = -\frac{v}{4\kappa^2} \text{ Tr} \left( [\tilde{X}_a, \tilde{X}_b] - i f_{ab}^c \tilde{X}_c \right)^2 \]

Dimensional reduction of YM action.

\[ \tilde{X}_a = \tilde{L}_a \] is a classical solution.
Large-N reduction works on group manifolds.

Matrix model

\[ S = -\frac{\nu}{4\kappa^2} \text{Tr} \left( \left[ \hat{X}_a, \hat{X}_b \right] - i f_{abc} \hat{X}_c \right)^2 \]

is equivalent to YM theory on G, if we expand \( \hat{X}_a \) around \( \hat{L}_a = T_a^{(\text{reg})} \otimes 1_k \), and take the large-N limit.
Gauge theories on group manifolds (cont’d)

The same redefinition holds for the matter fields of adjoint representation. The resultant theory is also the dimensional reduction to zero-dim.

\[ \frac{v}{\kappa^2} \text{Tr} \left( -\frac{1}{2} [\tilde{L}_a + \tilde{X}_a, \tilde{\phi}]^2 + V(\tilde{\phi}) \right) \]

\[ \rightarrow \frac{v}{\kappa^2} \text{Tr} \left( -\frac{1}{2} [\tilde{X}_a, \tilde{\phi}]^2 + V(\tilde{\phi}) \right) \]

Gauge symmetry: \[ \tilde{X}'_a = \hat{U} \tilde{X}_a \hat{U}^\dagger, \quad \tilde{\phi}' = \hat{U} \tilde{\phi} \hat{U}^\dagger \text{ etc.} \]

- If G is semi-simple, no massless mode exists.
- The background is stable at least perturbatively.
- Probably tunneling to the other solutions is suppressed in \( k \rightarrow \infty \) limit.

\[ \rightarrow \] No need of remedies such as quenching or twisting.

Regularization that preserves gauge symmetry, SUSY, and \( G \times G \).
Large $N$ reduction for $N=4$ SYM on $R \times S^3$ and the AdS/CFT duality
N=4 SYM on RxS³

Equivalent to N=4 SYM on R⁴ by conformal mapping.

\[
S = \frac{1}{4\kappa^2} \int dt dg \, \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_{\mu} X_m D_{\mu} X_m + \frac{1}{8} X_m^2 - \frac{1}{4} [X_m, X_n]^2 \\
+ \frac{1}{2} \psi^\dagger D_t \psi + \frac{i}{2} \psi^\dagger \gamma^a e^a_{\mu} D_{\mu} \psi - \frac{1}{2} \psi^\dagger \gamma^m [X_m, \psi] \right)
\]

\[\mu = \theta, \varphi, \psi, \quad \hat{\mu}, \hat{\nu} = t, \theta, \varphi, \psi, \quad m, n = 4, \ldots, 9\]

PSU(2,2|4) symmetry (32 supercharges).

Express space derivatives in \( L \).

Reduced model (Time remains.)

\[
S_r = \frac{\nu}{\kappa^2} \int dt \, \text{Tr} \left[ \frac{1}{2} (D_t X_M)^2 - \frac{1}{4} [X_M, X_N]^2 + \frac{1}{2} \psi^\dagger D_t \psi - \frac{1}{2} \psi^\dagger \gamma^M [X_M, \psi] \\
+ \frac{1}{2} (X_a)^2 + \frac{1}{8} (X_m)^2 + i \epsilon_{abc} X_a X_b X_c + \frac{3i}{8} \psi \gamma^{123} \psi \right]
\]

The same form as plane wave matrix model (Berenstein-Maldacena-Nastase).

SU(2|4) symmetry (16 supercharges) is manifest.
Many explicit checks for perturbation series have been done for YM and CS theory on $S^3$. Ishiki-Shimasaki-Tsuchiya (08~09)

**N=4 SYM on R×S^3 (cont’d)**

BMN matrix model becomes equivalent to the N=4 SYM on R×S^3 in the large-N limit, if we expand $X_a$ ($a = 1 \sim 3$) around

$$L_a = \begin{pmatrix} L_a^{[0]} \\ L_a^{[1/2]} \otimes 1_2 \\ \vdots \\ L_a^{[K]} \otimes 1_{2^K+1} \end{pmatrix} \otimes 1_k$$

This regularization preserves SU(2) x SU(2|4) (16 supercharges).

Many explicit checks for perturbation series have been done for YM and CS theory on $S^3$. Ishiki-Shimasaki-Tsuchiya (08~09)
Generalizations to CS theory and Coset spaces
Chern-Simons-like theories on group manifolds

\[ S = \frac{1}{\omega} \int \operatorname{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \wedge *f \]

\[ f = f_{abc} e^a \wedge e^b \wedge e^c \]

\[ *f \in H^\text{dim} G - 3(G) \]

Gauge transformation

\[ S' = S - \frac{1}{3\omega} \int \operatorname{Tr} \left( U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU \right) \wedge *f \]

\[ = S - \frac{1}{3\omega} \int_{C_3} \operatorname{Tr} \left( U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU \right) \]

\[ = S + 2\pi n \]

\( \omega \) is appropriately chosen.

Reduced model

\[ S_r = \frac{v}{6\omega} f_{abc} \operatorname{Tr} \left( \frac{1}{2} f_{bcd} \hat{X}_a \hat{X}_d + \frac{2i}{3} \hat{X}_a \hat{X}_b \hat{X}_c \right) \]

Express derivatives in \( L \).

Set \( L = 0 \) formally.
Large N reduction on coset spaces

H: Subgroup of G. \[ a = (A, \alpha) \]

\[ H \\
G/H \]

In field theory, we can start from a theory on G, and apply a consistent reduction to G/H by imposing a constraint \( \delta_A \phi = 0 \), which is also right G invariant.

\( \delta_A \): infinitesimal left translation along A

For scalar field, \( \delta_A \phi = L_A \phi \).

Reduced model given by

\[
S = -\nu Tr \left( \frac{1}{2} \left[ \hat{L}_\alpha, \hat{\phi} \right]^2 + \frac{1}{2} m^2 \hat{\phi}^2 \right)
\]

with constraints \( [\hat{L}_A, \hat{\phi}] = 0 \)

is equivalent to the scalar field on G/H in the large-N limit.
Large N reduction on coset spaces (cont’d)

Similar construction works for gauge theory.

For vector field, \( \delta_A X_\alpha = L_A X_\alpha + f_{A\alpha\beta} X_\beta \).

Large-N YM theory on G/H is described by

\[
S = -\frac{\nu}{4\kappa^2} Tr \left( \left[ \hat{X}_\alpha, \hat{X}_\beta \right] - i f_{\alpha\beta\gamma} \hat{X}_\gamma - i f_{\alpha\beta A} \hat{L}_A \right)^2,
\]

with constraints \( \delta_A \hat{X}_\alpha = \left[ \hat{L}_A, \hat{X}_\alpha \right] - i f_{A\alpha\beta} \hat{X}_\beta = 0 \),

where \( \hat{L}_a \) is the same as in the gauge theory on G.

D-dim manifold is described by D matrices.

**Example**  
Gauge theory on \( S^4 = SO(5)/SO(4) \).  
Recovers \( R^4 \) in the infinite volume limit.
Summary and outlook
Summary

- The large N reduction holds on group manifolds and coset spaces.
- Background is stable at least perturbatively, if the group is semi-simple.
- Super symmetry is maintained at least partially. It will be useful as a tool for numerical analyses.

Outlook

- Generalization to arbitrary manifolds.
- Numerical simulation for N=4 SYM, for example.
- It might shed a light on matrix models for string theory, especially on the emergence of space-times.