Large N reduction on group manifolds and coset spaces

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What I would like to show:

"Large-N reduction works for group manifolds in its original form."

Matrix model

$$S = -\frac{\upsilon}{4\kappa^2} Tr\left(\left[\hat{X}_a, \hat{X}_b\right] - i f_{abc}\hat{X}_c\right)^2,$$

f_{abc} : structure constant of a group G

is equivalent to YM theory on G, if we expand $\hat{X}_a$ around *a special minimum* of S, and take the large-N limit.

Minimum of S: $\left[\hat{X}_a, \hat{X}_b\right] = i f_{abc}\hat{X}_c \;\Rightarrow\; \hat{X}_a$ are rep. matrices of G.

We consider a special representation of the form $\hat{L}_a = T_a^{(reg)} \otimes 1_k$,

where $T_a^{(reg)}$ is the rep. matrix for the regular representation of G , and 1_k is k x k unit matrix.

$T_a^{(reg)}$ is first truncated to n dims, and take the limit $n, k \to \infty$.

NB Different from fuzzy manifolds.

D-dimensional space-time emerges from D matrices.

Introduction

- Large N reduction
  large-N gauge theory is equivalent to lower dimensional models obtained by dimensional reduction.

- Conceptual importance: Emergence of space-time from matrix degrees of freedom.
- Practical use: Non-perturbative formulation of large N gauge theory.
  In particular, super symmetric Yang Mills theory.

- So far it has been investigated mainly on flat space-time.

- Generalization to S^3 was done. Ishii-Ishiki-Shimasaki-Tsuchiya. ('08)
  They have considered N=4 SYM on R x S^3 .

- General curved spacetime?
  Description of curved space-times by matrices. Hanada-Kawai-Kimura('06)
  Fluctuation around them is still not clear.

- Here we show that the large N reduction works on group manifolds and coset spaces.
  Usually large N reduction is shown in momentum space.
  We reconsider it in coordinate space to make the generalization easier.
  We consider a kind of bi-local field theory.

Outline

1. Introduction
2. Bi-local field theory interpretation of reduced model
3. Large N reduction on group manifolds
4. Large N reduction for N=4 SYM on $R \times S^3$
5. Summary and outlook

Bi-local field theory interpretation of reduced model

phi^3 matrix field theory on R^d

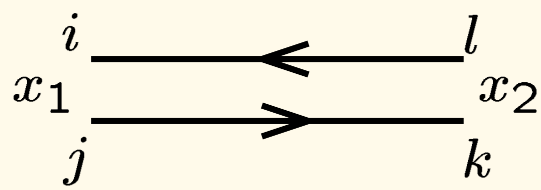
Action

$$S = \int d^dx \ \mathrm{Tr}\left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2}m^2\phi(x)^2 + \frac{1}{3}\kappa\phi(x)^3\right)$$

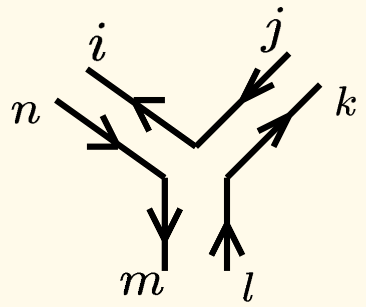
$\phi(x)$: N x N hermitian matrix

Propagator

$$\langle \phi(x_1)_{ij}\phi(x_2)_{kl}\rangle = D(x_1 - x_2)\delta_{il}\delta_{jk}$$



Vertex



$$-\kappa \ \delta_{ij}\delta_{kl}\delta_{mn}$$

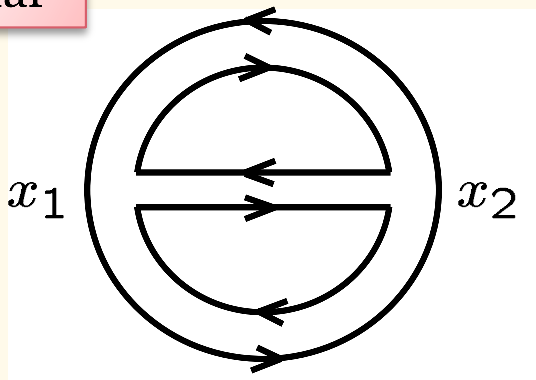
Large N limit

$$N \to \infty, \quad \kappa \to 0 \quad \text{with} \quad \kappa^2 N = \lambda \ \text{ fixed}$$

phi^3 matrix field on R^d (cont'd)

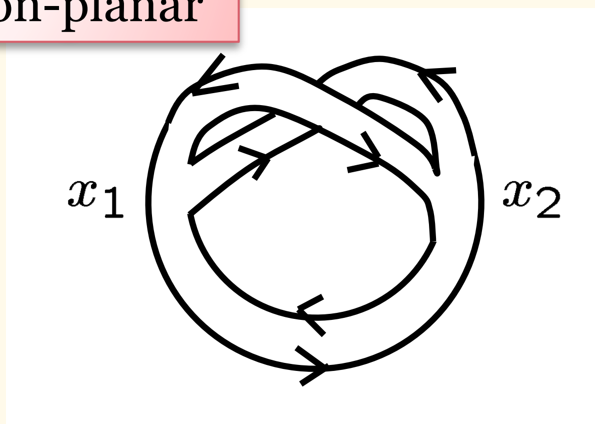
Free energy at the two-loop level

Planar



$$= \frac{1}{6} N^2 \lambda \int d^d x_1 d^d x_2 \; D(x_1 - x_2)^3$$

Non-planar



$$= \frac{1}{6} \lambda \int d^d x_1 d^d x_2 \; D(x_1 - x_2)^3$$

Suppressed

Large N reduction

$$S = \int d^d x \ \mathrm{Tr}\left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2}m^2\phi(x)^2 + \frac{1}{3}\kappa\phi(x)^3\right)$$

reduced model

Construct a matrix model by the following procedure:

$$\phi(x) \to \hat{\phi}, \quad \partial_\mu \to [i\hat{P}_\mu, \ ], \quad \int d^d x \to v$$

$$\Rightarrow \boxed{S_r = v\,\mathrm{Tr}\left(\frac{1}{2}[i\hat{P}_\mu, \hat{\phi}]^2 + \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{3}\kappa\hat{\phi}^3\right)}$$

More concretely,

$\hat{\phi}$: hermitian operator acting on the function space on R^d

$$\hat{P}_\mu|x\rangle = -\frac{1}{i}\frac{\partial}{\partial x^\mu}|x\rangle, \qquad \langle x|\hat{P}_\mu = \frac{1}{i}\frac{\partial}{\partial x^\mu}\langle x|$$

$|x\rangle \ (x \in R^d)$: coordinate basis.

Large N reduction (cont'd)

IR and UV cut off

Assume the volume of space-time is V.

Function space on space-time $\Rightarrow$ N dimensional vector space.

Space-time is divided into N cells of volume $v = V/N$.

Cut off momentum is given by $v = \left(\frac{2\pi}{\Lambda}\right)^d$.

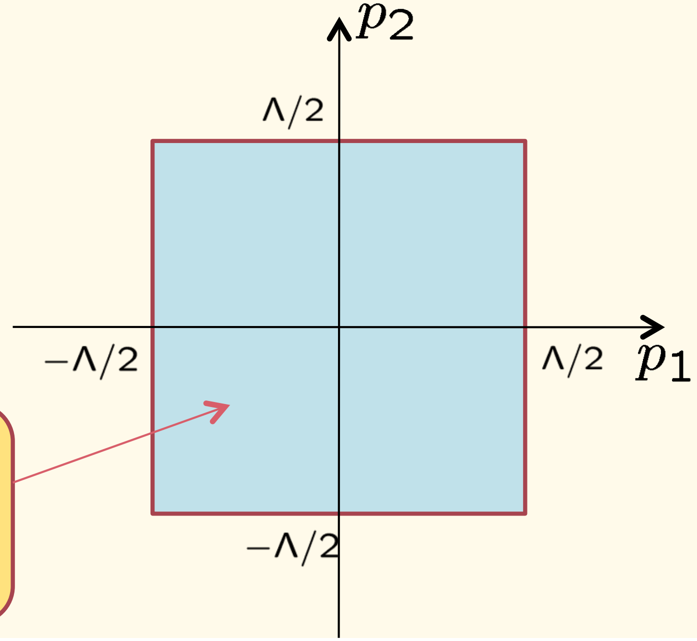
Ordinary proof

Momentum representation.

$$\hat{P}_\mu = \begin{pmatrix} p_\mu^{(1)} & & & \\ & p_\mu^{(2)} & & \\ & & \ddots & \\ & & & p_\mu^{(N)} \end{pmatrix}$$

$\hat{\phi}$: N x N hermitian matrix.

$(p_1^{(i)}, p_2^{(i)})$
Uniform distribution



Reduced model as a bi-local field theory

Bi-local field theory

Coordinate representation.

$$S_r = v\mathrm{Tr}\left(\frac{1}{2}[i\widehat{P}_\mu, \widehat{\phi}]^2 + \frac{1}{2}m^2\widehat{\phi}^2 + \frac{1}{3}\kappa\widehat{\phi}^3\right)$$

$$\boxed{\langle x|\widehat{\phi}|x'\rangle \equiv \phi(x, x')}$$

Bi-local field.

$$\widehat{P}_\mu|x\rangle = -\frac{1}{i}\frac{\partial}{\partial x^\mu}|x\rangle, \qquad \langle x|\widehat{P}_\mu = \frac{1}{i}\frac{\partial}{\partial x^\mu}\langle x|$$

$$\boxed{\begin{aligned} S_r = v\int d^dx d^dx' \left(-\frac{1}{2}\phi(x', x)\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu}\right)^2 \phi(x, x') + \frac{1}{2}m^2\phi(x', x)\phi(x, x')\right) \\ + v\int d^dx d^dx' d^dx'' \frac{1}{3}\kappa_r \phi(x, x')\phi(x', x'')\phi(x'', x) \end{aligned}}$$

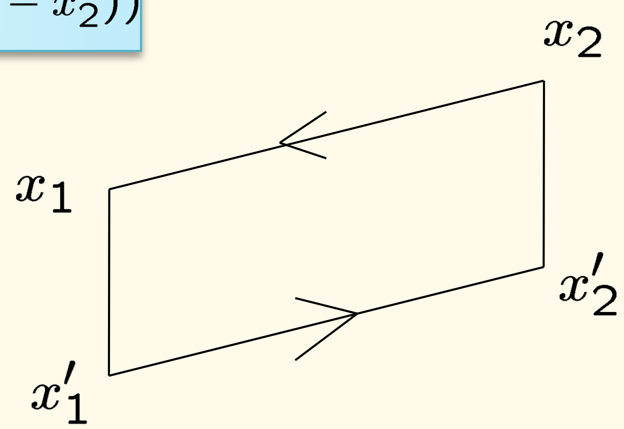
Change of variables

$$\boxed{X^\mu = x^\mu, \quad \xi^\mu = x^\mu - x'^\mu} \Rightarrow \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu}\right)\phi(x, x') = \frac{\partial}{\partial X^\mu}\phi(x, x')$$

Perturbative expansion in cordinate space

Propagator

$$\langle \phi(x_1, x_1')\phi(x_2', x_2)\rangle = \frac{1}{v} D(x_1 - x_2)\delta^d((x_1 - x_1') - (x_2 - x_2'))$$



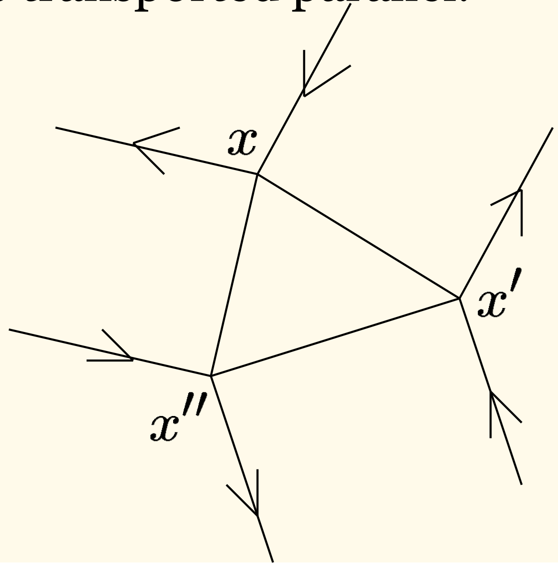
End points propagate like particles.

Relative coordinate $x_1 - x_1'$ conserves.

$$x_1 - x_2 = x_1' - x_2'$$

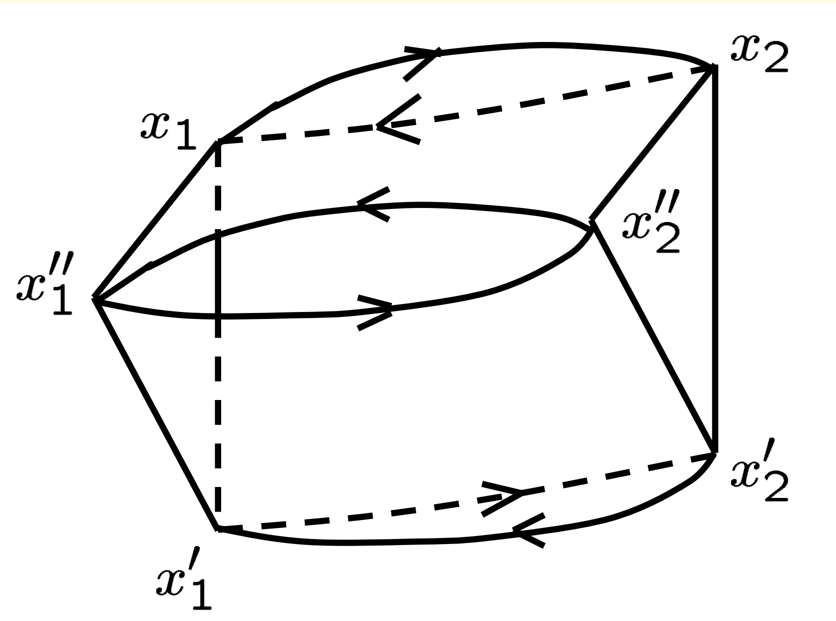
End points are transported parallel.

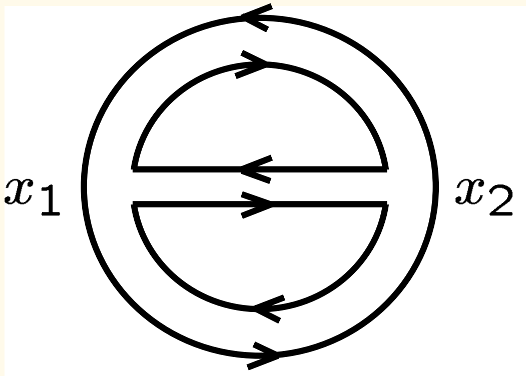
Vertex



Free energy at the two-loop level

Planar





$$\frac{1}{6}N^2\lambda\int d^dx_1 d^dx_2\ D(x_1-x_2)^3$$

$$\times\frac{1}{N^2V}$$

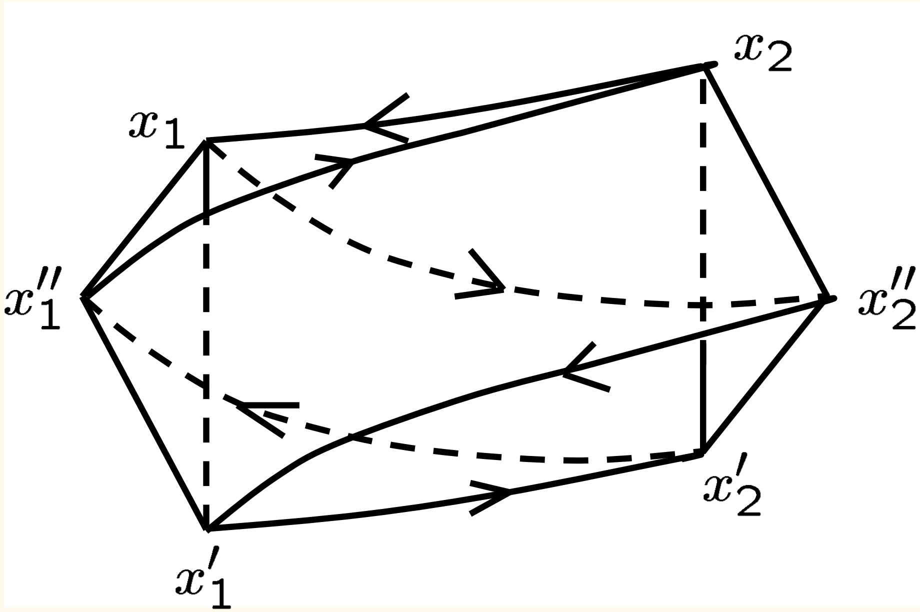
$$\frac{\kappa^2}{6v}\delta^d(0)\underline{V^2}\int d^dx_1 d^dx_2\ D(x_1-x_2)^3$$

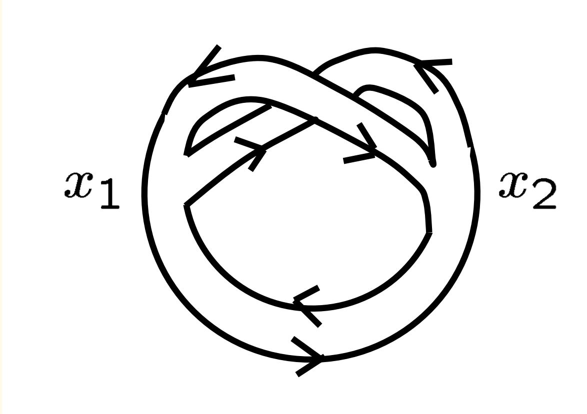
$$\times\frac{1}{N^2v}$$

$$V=Nv \qquad \delta^d(0)=\frac{1}{v}$$

Free energy at the two-loop level (cont'd)

Non-planar





$$\frac{1}{6}\lambda \int d^dx_1 d^dx_2 \; D(x_1 - x_2)^3$$

$$\frac{\kappa^2}{6v}\delta^d(0)\int d^dx_1 d^dx_1' d^dx_2 d^dx_2'' D(x_1 - x_2) D(x_1' - x_2'') D(x_1 - x_2'')$$

$$\times \frac{1}{N^2 v} \quad \neq \quad \times \frac{1}{N^2 V}$$

Suppressed by factor $1/V^2$ compared with planar diagrams.

$$\frac{\kappa^2}{6v}\delta^d(0) V^2 \int d^dx_1 d^dx_2 \; D(x_1 - x_2)^3$$

Correspondence between reduced model and original theory

Limit in reduced model

$$N \to \infty, \quad \kappa \to 0, \quad v \to 0 \quad \text{with} \quad V = Nv \to \infty, \quad \lambda = \kappa^2 N \ \text{ fixed}$$

Reduced model reproduces original field theory.

Free energy:

$$\frac{F}{N^2 V} = \frac{F_r}{N^2 v}$$

Correlation functions:

$$\frac{1}{N^{q/2+1}} \langle \mathrm{Tr}(\phi(x_1)\phi(x_2)\cdots\phi(x_q))\rangle = \frac{1}{N^{q/2+1}} \langle \mathrm{Tr}(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_q))\rangle_r$$

$$\hat{\phi}(x) = e^{i\hat{P}_\mu x^\mu} \hat{\phi} e^{-i\hat{P}_\nu x^\nu}$$

Large N reduction on Torus T^d

Torus has finite volume V. $\Longrightarrow$ No $1/V$ suppression.

Introduce *another* index in order to suppress non-planar diagrams.

$\hat{\phi}$ Matrix valued bi-local operator:

$$\phi(x,x') \to \phi(x,x')_{\alpha\beta} \quad (\alpha,\beta = 1,\cdots,k)$$

$$\hat{P}_\mu \to \hat{P}_\mu \otimes 1_k$$

Function space on the torus $\Longrightarrow$ n-dim vector space.

Dimension of $\hat{\phi}$ is N=nk.

Large-N limit of the reduced model:

$$n \to \infty, \quad k \to \infty, \quad \kappa \to 0, \quad \text{with} \;\; \lambda = \kappa^2 N = \kappa^2 nk \;\; \text{fixed}$$

$\Longrightarrow$ Non-planar diagrams are suppressed by $1/k^2$, and the reduced model reproduces the original field theory in the planar limit.

Large N reduction for gauge theory

Apply the rule to the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \Rightarrow i[\hat{P}_\mu + \hat{A}_\mu, \hat{P}_\nu + \hat{A}_\nu] = i[\hat{X}_\mu, \hat{X}_\nu]$$

$$\begin{cases} \partial_\mu \to [i\hat{P}_\mu, \ \ ] \\ A_\mu \to \hat{A}_\mu \end{cases}$$

$$\boxed{\hat{X}_\mu = \hat{P}_\mu + \hat{A}_\mu}$$

Reduced model of YM theory

$$S'_r = -\frac{v}{4\kappa^2}\mathrm{Tr}[\hat{X}_\mu, \hat{X}_\nu]^2$$

Dimensional reduction of YM theory to zero dimension.

$\hat{P}_\mu$ is interpreted as a background of $\hat{X}_\mu$.

This background is unstable because of massless modes.

$\Rightarrow$ Quenching, Twisting,... Not consistent with SUSY.

Bhanot-Heller-Neuberger ('82) Gross-Kitazawa ('82)

Large N reduction on group manifolds

Notes on group manifolds

Lie group

G: compact and connected Lie group. (Later we will assume G is semi-simple.)

$t_a \quad (a = 1, \cdots, \dim G)$: Generators of G. $\qquad [t_a, t_b] = i f_{ab}{}^c t_c$

$|g\rangle \quad (g \in G)$: Coordinate basis of the function space on G.

Left and right translations

Left translation: $\hat{U}_L(h)|g\rangle = |hg\rangle, \quad \langle g|\hat{U}_L(h) = \langle h^{-1}g|$

Right translation: $\hat{U}_R(h)|g\rangle = |gh^{-1}\rangle, \quad \langle g|\hat{U}_R(h) = \langle gh|$

$h \in G$

For a function on G $\psi(g) = \langle g|\psi\rangle$,

$$(\hat{U}_L(h)\psi)(g) = \psi(h^{-1}g), \quad (\hat{U}_R(h)\psi)(g) = \psi(gh)$$

Notes on group manifolds (cont'd)

Killing vectors

Right invariant Killing vector $\hat{L}_a$: $e^{i\epsilon \hat{L}_a} = \hat{U}_L(e^{i\epsilon t_a})$ infinitesimal left translation

Left $\hat{K}_a$: $e^{i\epsilon \hat{K}_a} = \hat{U}_R(e^{i\epsilon t_a})$

right

Comm. Rel. $[\hat{L}_a, \hat{L}_b] = i f_{ab}{}^c \hat{L}_c, \quad [\hat{K}_a, \hat{K}_b] = i f_{ab}{}^c \hat{K}_c, \quad [\hat{L}_a, \hat{K}_b] = 0$

In terms of differential operators,

$$\hat{L}_a|g\rangle = -\mathcal{L}_a|g\rangle, \quad \langle g|\hat{L}_a = \mathcal{L}_a\langle g|,$$
$$\hat{K}_a|g\rangle = -\mathcal{K}_a|g\rangle, \quad \langle g|\hat{K}_a = \mathcal{K}_a\langle g|$$

Group version of $\hat{P}_\mu|x\rangle = -\frac{1}{i}\frac{\partial}{\partial x^\mu}|x\rangle, \quad \langle x|\hat{P}_\mu = \frac{1}{i}\frac{\partial}{\partial x^\mu}\langle x|$.

Notes on group manifolds (cont'd)

Invariant 1-forms

$$\begin{cases} \text{Right inv. 1- form} & e^a \\ \text{Left} & s^a \end{cases} \quad \Longleftarrow \quad d = dx^\mu \frac{\partial}{\partial x^\mu} = ie^a \mathcal{L}_a = is^a \mathcal{K}_a$$

Maurer-Cartan equation $\quad de^a - \frac{1}{2} f_{bc}{}^a e^b \wedge e^c = 0, \quad ds^a - \frac{1}{2} f_{bc}{}^a s^b \wedge s^c = 0$

Right and left invariant metric

$$h_{\mu\nu} = e^a_\mu e^a_\nu = s^a_\mu s^a_\nu$$

Haar measure

$$dg = d^{\dim G} x \sqrt{h} = e^1 \wedge e^2 \wedge \cdots \wedge e^{\dim G}$$ Left right invariant.

volume $\quad V = \int dg$

phi^3 matrix scalar field theory on G

Scalar phi^3 theory on G

$$h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -(\mathcal{L}_a\phi)^2$$

$$\boxed{S = \int dg \ \mathrm{Tr}\left(-\frac{1}{2}(\mathcal{L}_a\phi(g))^2 + \frac{1}{2}m^2\phi(g)^2 + \frac{1}{3}\kappa\phi(g)^3\right)}$$

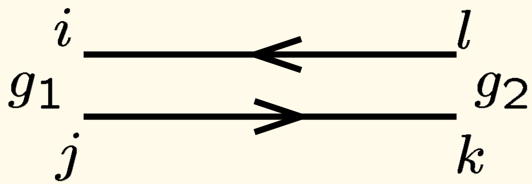
$\phi(g)$: N x N hermitian, each element is a function on G.

Possesses G x G symmetry.

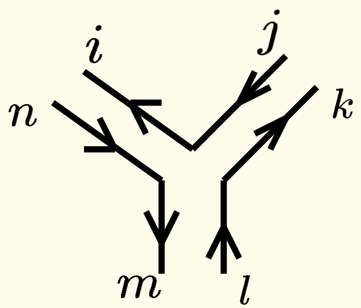
Propagator

Right G invariance

$$\langle\phi(g_1)_{ij}\phi(g_2)_{kl}\rangle = \Delta(g_1 g_2^{-1})\delta_{il}\delta_{jk}$$



Vertex



$$-\kappa \ \delta_{ij}\delta_{kl}\delta_{mn}$$

Large N reduction on G

$$S = \int dg \ \mathrm{Tr}\left(-\frac{1}{2}(\mathcal{L}_a\phi(g))^2 + \frac{1}{2}m^2\phi(g)^2 + \frac{1}{3}\kappa\phi(g)^3\right)$$

Reduced model

Construct a matrix model by the following procedure:

$$\phi(g) \to \hat{\phi}, \quad \mathcal{L}_a \to [\hat{L}_a \otimes 1_k, \ ], \quad \int dg \to v$$

Consider the tensor product of the function space on G
and a k-dim vector space,
$\hat{\phi}$: hermitian operator acting on the tensor space.

$$\Rightarrow \quad S_r = v\mathrm{Tr}\left(-\frac{1}{2}[\hat{L}_a, \hat{\phi}]^2 + \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{3}\kappa\hat{\phi}^3\right)$$

Reduced model as a bi-local field theory

Bi-local field theory Coordinate representation.

$$S_r = v\mathrm{Tr}\left(-\frac{1}{2}[\hat{L}_a,\hat{\phi}]^2 + \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{3}\kappa\hat{\phi}^3\right)$$

$\boxed{\langle g|\hat{\phi}|g'\rangle \equiv \phi(g,g')}$: bi-local kxk matrix field on G

$$\hat{L}_a|g\rangle = -\mathcal{L}_a|g\rangle, \quad \langle g|\hat{L}_a = \mathcal{L}_a\langle g|$$

$$\Downarrow$$

$$\begin{aligned} S_r = v\int dg dg'\ \mathrm{tr}\left\{\frac{1}{2}\phi(g',g)\left(\mathcal{L}_a^{(g)} + \mathcal{L}_a^{(g')}\right)^2\phi(g,g') + \frac{1}{2}m^2\phi(g',g)\phi(g,g')\right\} \\ + v\int dg dg' dg''\ \frac{1}{3}\kappa\ \mathrm{tr}(\phi(g,g')\phi(g',g'')\phi(g'',g)) \end{aligned}$$

Change of variables

$$u = g, \quad \zeta = g'^{-1}g \quad \Longrightarrow \quad \left(\mathcal{L}_a^{(g)} + \mathcal{L}_a^{(g')}\right)\phi(g,g') = \mathcal{L}_a^{(u)}\phi(g,g')$$

Haar measure is invariant.

Perturbative expansion

Propagator

$$\langle \phi(g_1, g_1')_{\alpha\beta} \phi(g_2', g_2)_{\gamma\delta} \rangle = \frac{1}{v} \Delta(g_1 g_2^{-1}) \delta({g_1'}^{-1} g_1, {g_2'}^{-1} g_2) \delta_{\alpha\delta} \delta_{\beta\gamma}$$

End points propagate as particles on G.
The relative coordinate conserves during propagation.
The situation is the same as in the flat space, and the same analysis holds.

$\Rightarrow$ Large N reduction holds on G.

All we need is the right G invariance.

⇔ Action is written in terms of left derivatives.

UV regularization

The function space on G is identified with the representation space of the regular representation.

$$(\hat{U}_L(h)\psi)(g) = \psi(h^{-1}g), \quad (\hat{U}_R(h)\psi)(g) = \psi(gh)$$

Peter-Weyl's theorem $\psi(g) = \sum_r \sum_{ij} c_{ij}^{[r]} R_{ij}^{[r]}(g)$ r runs for all irreducible representations.

$$V_{reg} = \bigoplus_r V_r \otimes V_{r^*}$$

$$\hat{L}_a = \bigoplus_r L_a^{[r]} \otimes 1_{d_r}, \quad \hat{K}_a = \bigoplus_r 1_{d_r} \otimes L_a^{[r^*]}$$

$L_a^{[r]}$: rep. matrix of t_a in the rep. r

d_r : dimension of the rep. r

Corresponding to UV cut off Λ , introduce $I_\Lambda = \{r; C_2(r) < \Lambda^2\}$.

Restrict the sum to I_Λ .

Preserves G x G symmetry.

Correspondence between reduced model and original theory

$$S_r = v \mathrm{Tr}\left(-\frac{1}{2}[\hat{L}_a, \hat{\phi}]^2 + \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{3}\kappa\hat{\phi}^3\right)$$

$\hat{\phi}$: NxN hermitian matrix

$$\hat{L}_a = \left(\bigoplus_{r\in I_\Lambda} L_a^{[r]} \otimes 1_{d_r}\right) \otimes 1_k$$

G x G invariance

parameters

- $n = \sum_{r\in I_\Lambda} d_r^2$ Function space on G $\sim$ n-dim vector space
- $N = nk$ Size of the matrix
- $v = V/n$ Volume of each cell

limit $\Lambda \to \infty \ (n \to \infty), \ \ k \to \infty, \ \ \kappa \to 0, \ \ \text{with } \lambda = \kappa^2 N = \kappa^2 nk \ \text{ fixed}$

The reduced model reproduces the original theory in the planar limit.

Correlation functions

$$\frac{1}{N^{q/2+1}}\langle \mathrm{Tr}(\phi(x_1)\phi(x_2)\cdots\phi(x_q))\rangle = \frac{1}{N^{q/2+1}}\langle \mathrm{Tr}(\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_q))\rangle_r$$

$$\hat{\phi}(g) = e^{i\theta^a\hat{L}_a}\hat{\phi}e^{-i\theta^b\hat{L}_b} \ \text{for} \ \ g = e^{i\theta^a t_a}.$$

Example: G=SU(2)=S^3

$$L_a = \begin{pmatrix} L_a^{[0]} & & & \\ & L_a^{[1/2]} \otimes 1_2 & & \\ & & \ddots & \\ & & & L_a^{[K]} \otimes 1_{2K+1} \end{pmatrix} \otimes 1_k$$

$$n = \sum_{j=0}^{K} (2j+1)^2$$

$$v = 16\pi^2/n$$

$$N = nk$$

Preserves SO(4)=SU(2)xSU(2) symmetry.

Gauge theories on group manifolds

Expand gauge fields by the right invariant 1-form, and use Maurer-Cartan equation.

$$A = X_a e^a \Rightarrow F = dA + iA \wedge A$$

Spin connection

$$= \frac{1}{2}(i\mathcal{L}_a X_b - i\mathcal{L}_b X_a + f_{ab}{}^c X_c + i[X_a, X_b]) e^a \wedge e^b$$

YM action

$$S = \frac{1}{4\kappa^2}\int \mathrm{Tr}(F \wedge *F)$$

Ishii-Ishiki-Shimasaki-Tsuchiya ('08)

$$= -\frac{1}{4\kappa^2}\int dg\ \mathrm{Tr}(\mathcal{L}_a X_b - \mathcal{L}_b X_a - i f_{ab}{}^c X_c + [X_a, X_b])^2$$

Right G invariant !

Reduced model

$$S_r = -\frac{v}{4\kappa^2}\mathrm{Tr}([\hat{L}_a, \hat{X}_b] - [\hat{L}_b, \hat{X}_a] - i f_{ab}{}^c \hat{X}_c + [\hat{X}_a, \hat{X}_b])^2$$

$$= -\frac{v}{4\kappa^2}\mathrm{Tr}([\hat{L}_a + \hat{X}_a, \hat{L}_b + \hat{X}_b] - i f_{ab}{}^c(\hat{L}_c + \hat{X}_c))^2$$

Absorb the background L to X.

$$\boxed{S'_r = -\frac{v}{4\kappa^2}\mathrm{Tr}\left([\hat{X}_a, \hat{X}_b] - i f_{ab}{}^c \hat{X}_c\right)^2}$$

Dimensional reduction of YM action.

$\hat{X}_a = \hat{L}_a$ is a classical solution.

Gauge theories on group manifolds (cont'd)

Large-N reduction works on group manifolds.

Matrix model

$$S=-\frac{\upsilon}{4\kappa^2}Tr\left(\left[\hat{X}_a,\hat{X}_b\right]-i f_{abc}\hat{X}_c\right)^2$$

is equivalent to YM theory on G, if we expand $\hat{X}_a$ around $\hat{L}_a=T_a^{(reg)}\otimes 1_k$,

and take the large-N limit.

Gauge theories on group manifolds (cont'd)

The same redefinition holds for the matter fields of adjoint representation. The resultant theory is also the dimensional reduction to zero-dim.

$$\frac{v}{\kappa^2}\mathrm{Tr}\left(-\frac{1}{2}[\hat{L}_a+\hat{X}_a,\hat{\phi}]^2+V(\hat{\phi})\right)$$
$$\to \frac{v}{\kappa^2}\mathrm{Tr}\left(-\frac{1}{2}[\hat{X}_a,\hat{\phi}]^2+V(\hat{\phi})\right)$$

Gauge symmetry $\hat{X}'_a=\hat{U}\hat{X}_a\hat{U}^\dagger,\ \ \hat{\phi}'=\hat{U}\hat{\phi}\hat{U}^\dagger$ etc.

If G is semi-simple, no massless mode exists.
The background is stable at least perturbatively.
Probably tunneling to the other solutions is suppressed in $k\to\infty$ limit.

⇒ No need of remedies such as quenching or twisting.

⇒ Regularization that preserves gauge symmetry, SUSY, and G x G.

Large N reduction for N=4 SYM on $R \times S^3$ and the AdS/CFT duality

N=4 SYM on RxS^3

Equivalent to N=4 SYM on R^4 by conformal mapping.

$$S=\frac{1}{4\kappa^2}\int dtdg\,\mathrm{Tr}\left(\frac{1}{4}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}}+\frac{1}{2}D_{\hat{\mu}}X_mD^{\hat{\mu}}X_m+\frac{1}{8}X_m^2-\frac{1}{4}[X_m,X_n]^2\right.$$
$$\left.+\frac{1}{2}\Psi^\dagger D_t\Psi+\frac{i}{2}\Psi^\dagger\gamma^a e_a^\mu D_\mu\Psi-\frac{1}{2}\Psi^\dagger\gamma^m[X_m,\Psi]\right)$$

$$\mu=\theta,\varphi,\psi,\quad \hat{\mu},\hat{\nu}=t,\theta,\varphi,\psi,\quad m,n=4,\cdots,9$$

PSU(2,2|4) symmetry (32 supercharges).

Express space derivatives in **L**.

Set **L** =0 formally.

Reduced model (Time remains.)

$$\boxed{S_r=\frac{v}{\kappa^2}\int dt\,\mathrm{Tr}\left[\frac{1}{2}(D_tX_M)^2-\frac{1}{4}[X_M,X_N]^2+\frac{1}{2}\Psi^\dagger D_t\Psi-\frac{1}{2}\Psi^\dagger\gamma^M[X_M,\Psi]+\frac{1}{2}(X_a)^2+\frac{1}{8}(X_m)^2+i\epsilon_{abc}X_aX_bX_c+\frac{3i}{8}\Psi^\dagger\gamma^{123}\Psi\right]}$$

The same form as plane wave matrix model (Berenstein-Maldacena-Nastase).

SU(2|4) symmetry (16 supercharges) is manifest.

N=4 SYM on RxS^3 (cont'd)

BMN matrix model becomes equivalent to the **N**=4 SYM on RxS^3 in the large-N limit, if we expand X_a $(a=1\sim 3)$ around

$$L_a = \begin{pmatrix} L_a^{[0]} & & & \\ & L_a^{[1/2]} \otimes 1_2 & & \\ & & \ddots & \\ & & & L_a^{[K]} \otimes 1_{2K+1} \end{pmatrix} \otimes 1_k$$

This regularization preserves SU(2) x SU(2|4) (16 supercharges).

Many explicit checks for perturbation series have been done for YM and CS theory on S^3. Ishiki-Shimasaki-Tsuchiya (08~09)

Generalizations to CS theory and Coset spaces

Chern-Simons-like theories on group manifolds

$$S = \frac{1}{\omega}\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \wedge *f$$

$$f = f_{abc}\, e^a \wedge e^b \wedge e^c$$

$$*f \in H^{\dim G - 3}(G)$$

Gauge transformation

$$\begin{aligned} S' &= S - \frac{1}{3\omega}\int \mathrm{Tr}\left(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU\right) \wedge *f \\ &= S - \frac{1}{3\omega}\int_{C_3} \mathrm{Tr}\left(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU\right) \qquad \text{Poincare duality} \\ &= S + 2\pi n \end{aligned}$$

ω is appropriately chosen.

Express derivatives in **L**.

Set **L** =0 formally.

Reduced model

$$S_r = \frac{v}{6\omega} f^{abc} \mathrm{Tr}\left(\frac{1}{2} f_{bcd} \hat{X}_a \hat{X}_d + \frac{2i}{3} \hat{X}_a \hat{X}_b \hat{X}_c\right)$$

Large N reduction on coset spaces

H: Subgroup of G. $a = (A, \alpha)$

H G/H

In field theory, we can start from a theory on G, and apply a consistent reduction to G/H by imposing a constraint $\delta_A \phi = 0$, which is *also right G invariant*.

δ_A :infinitesimal left translation along A

For scalar field, $\delta_A \phi = L_A \phi$.

Reduced model given by

$$S = -\upsilon Tr\left(\frac{1}{2}\left[\hat{L}_\alpha, \hat{\phi}\right]^2 + \frac{1}{2} m^2 \hat{\phi}^2\right)$$

with constraints $\left[\hat{L}_A, \hat{\phi}\right] = 0$

is equivalent to the scalar field on G/H in the large-N limit.

Large N reduction on coset spaces (cont'd)

Similar construction works for gauge theory.

For vector field, $\delta_A X_\alpha = L_A X_\alpha + f_{A\alpha\beta} X_\beta .$

Large-N YM theory on G/H is described by

$$S = -\frac{\upsilon}{4\kappa^2} Tr\left(\left[\hat{X}_\alpha, \hat{X}_\beta\right] - i f_{\alpha\beta\gamma}\hat{X}_\gamma - i f_{\alpha\beta A}\hat{L}_A\right)^2 ,$$

with constraints $\delta_A \hat{X}_\alpha = \left[\hat{L}_A, \hat{X}_\alpha\right] - i f_{A\alpha\beta}\hat{X}_\beta = 0,$

where $\hat{L}_a$ is the same as in the gauge theory on G.

D-dim manifold is described by D matrices.

<u>Example</u> Gauge theory on S^4=SO(5)/SO(4).
Recovers R^4 in the infinite volume limit.

Summary and outlook

Summary

- The large N reduction holds on group manifolds and coset spaces.
- Background is stable at least perturbatively, if the group is semi-simple.
- Super symmetry is maintained at least partially.

It will be useful as a tool for numerical analyses.

Outlook

- Generalization to arbitrary manifolds.
- Numerical simulation for N=4 SYM, for example.
- It might shed a light on matrix models for string theory, especially on the emergence of space-times.