Moduli Dynamics of Strings on AdS×S

Antal Jevicki

On the occasion of **Yoichi Kazama**'s 60^{th} Birthday

February 13, 2010

In collaboration with **I. Aniceto** and **K. Jin**

Overview of the talk:

- Classical string solutions: spiky strings
- AdS string as a sigma model
- N-spike string solutions
- Moduli dynamics
- Strings on $R\times S^2$: Giant magnons
- N-magnon dynamics
- Poisson structures and Quantization
- Conclusions and Future

I. Introduction

- Gauge/String duality:

  $\mathcal{N}$=4 SYM is given by string theory on $AdS_5 \times S^5$

  [Exact: Bethe Ansatz, all coupling, …]

- Classical strings:

  -- Offers predictions for strong coupling SYM

  -- Alday-Maldacena program of evaluating amplitudes in YM (minimal area surfaces)

  -- Study of classical string moduli space promises to give deep insight into AdS/CFT

Folded string and spiky extensions:

- Conformal gauge : $T_{++} = T_{--} = 0$
- Ansatz :

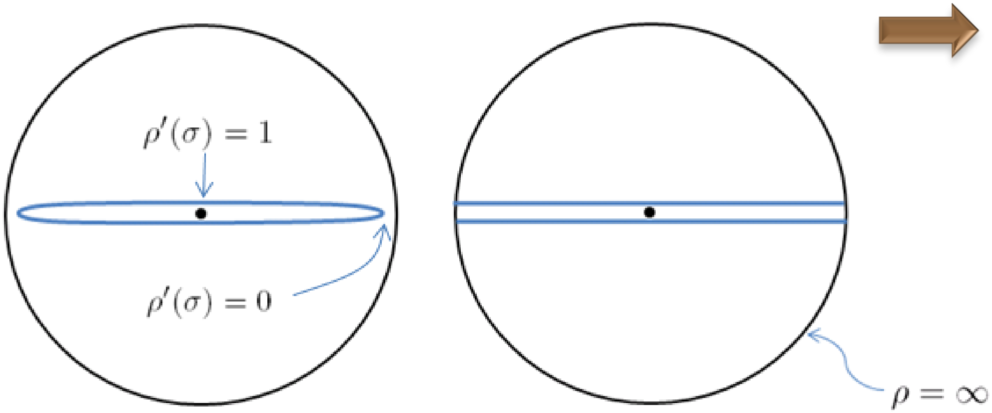
$$\begin{aligned} t &= c\,\tau \\ \theta &= c\,\omega\,\tau \\ \rho &= \rho\,(\sigma) \end{aligned}$$

GKP: hep-th/**0204051**

- Solution :

$$\boxed{\rho'^2(\sigma) = c^2(\cosh^2\rho - \omega^2\sinh^2\rho)}$$

$$\Rightarrow \rho(\sigma) = \mathrm{arccosh}(\mathrm{nd}(\omega\sigma, \frac{1}{\omega}))$$



(a) w>1

(b) w=1

$$\boxed{E - S = \frac{\sqrt{\lambda}}{\pi}\ln(\frac{S}{\sqrt{\lambda}}) + \cdots}$$

Static n-spike solution:

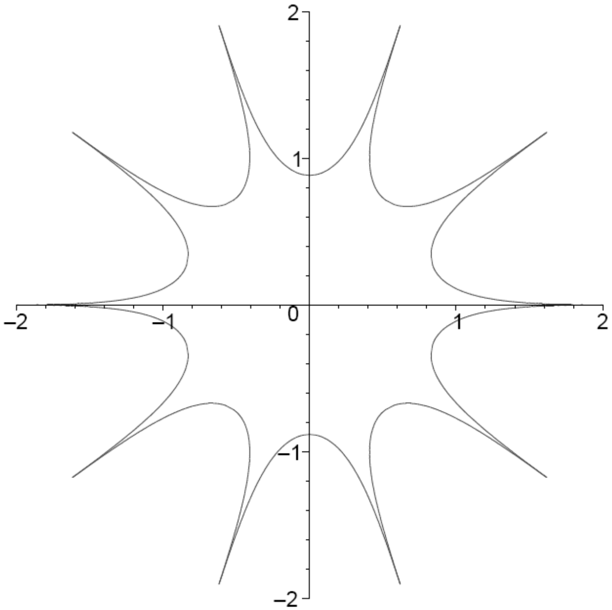
- Physical gauge with ansatz :

$$\begin{aligned} t &= \tau \\ \theta &= \omega\,\tau + \sigma \\ \rho &= \rho\,(\sigma) \end{aligned}$$

M. Kruczenski: hep-th/**0410226**

- Equation :

$$\rho'(\sigma) = \frac{1}{2}\frac{\sinh 2\rho}{\sinh 2\rho_0}\frac{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sqrt{\cosh^2\rho - \omega^2 \sinh^2\rho}}$$



$$\boxed{E - S = n\frac{\sqrt{\lambda}}{2\pi}\ln\frac{S}{\sqrt{\lambda}} + \cdots}$$

To study the dynamics of spikes, it is convenient to introduce the soliton picture.

Spikes as sinh-Gordon solitons:

- Asymptotics near the turning point: GKP solution

$$\rho'^2 = \cosh^2\rho - \omega^2\sinh^2\rho \sim \frac{1}{4}e^{2\rho}(1-\omega^2+(1+\omega^2)2e^{-2\rho})$$

- Let $\omega = 1+2\eta$ where $\eta \ll 1$, then one gets

$$\rho'^2 \sim e^{2\rho}(e^{-2\rho}-\eta)$$

- Denote $u = e^{-\rho}$, we have

$$u'^2 \sim u^2 - \eta$$

- It is easy to solve for

$$\rho(\sigma) = -\ln\sqrt{\eta}\cosh(\sigma-\sigma_0)$$

$$\Rightarrow \boxed{\alpha = \ln(2\rho'^2) = \ln(2\tanh^2(\sigma-\sigma_0))}$$

soliton

AJ, K. Jin, C. Kalousios, A. Volovich: 0712.1193

AdS string as a sigma model:

- We parameterize AdS_d with *d+1* embedding coordinates *Y* subject to the constraint

$$Y^2 = -Y_{-1}^2 - Y_0^2 + Y_1^2 + \cdots + Y_{d-1}^2 = -1$$

- Conformal gauge action :

$$A = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \Big(\partial Y \cdot \partial Y + \lambda(\tau, \sigma)(Y^2 + 1) \Big)$$

- Equations of motion :

$$Y_{\xi\eta} - (Y_\xi \cdot Y_\eta) Y = 0$$

- Virasoro constraints :

$$Y_\xi^2 = 0 \qquad Y_\eta^2 = 0$$

II. AdS$_3$ σ-model reduction:

- Done at the level of equations of motion:

K. Pohlmeyer, '76

- Choose a basis: $e_i = (Y, Y_\xi, Y_\eta, B_4)$, where B$_4$ is an orthonormal vector to the rest.
- Define: $\alpha(\xi,\eta) \equiv \ln[Y_\xi \cdot Y_\eta]$

$$u \equiv B_4 \cdot Y_{\xi\xi} \qquad v \equiv B_4 \cdot Y_{\eta\eta}$$

- EOMs: $\alpha_{\xi\eta} - e^{\alpha} - uve^{-\alpha} = 0$

$$u_\eta = 0 \qquad v_\xi = 0$$

Generalized sinh-Gordon equation!

H. J. de Vega and N. Sanchez, '93

Reconstruction of the string: Inverse scattering

- Lax pair:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha_\xi & 0 & u \\ e^\alpha & 0 & 0 & 0 \\ 0 & 0 & -ue^{-\alpha} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ e^\alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha_\eta & v \\ 0 & -ve^{-\alpha} & 0 & 0 \end{pmatrix}$$

- Zero curvature condition.
- Scattering equations:

$$(\partial_\xi - A_1)\phi = 0 \qquad (\partial_\xi - B_1)\psi = 0$$
$$(\partial_\eta - A_2)\phi = 0 \qquad (\partial_\eta - B_2)\psi = 0$$

- Construction of the string solution:

$$Z_1 \equiv Y_1 + iY_0 = \phi_1^*\psi_1 - \phi_2^*\psi_2 \qquad Z_2 \equiv Y_1 + iY_2 = \phi_2^*\psi_1^* - \phi_1^*\psi_2^*$$

One-spike dynamical solution:

$$\begin{aligned} Z_1 &= \frac{e^{i\tau}}{e^{(\sigma-\tau)/\tilde{v}_1} + e^{-(\sigma+\tau)\tilde{v}_1}} \Big\{ e^{(\sigma-\tau)/\tilde{v}_1} \cosh\sigma \\ &\quad + e^{-(\sigma+\tau)\tilde{v}_1} \frac{(1-i\tilde{v}_1)^2((1+\tilde{v}_1^2)\cosh\sigma + 2\tilde{v}_1 \sinh\sigma)}{1-\tilde{v}_1^4} \Big\}, \\ Z_2 &= \frac{-ie^{i\tau}}{e^{(\sigma-\tau)/\tilde{v}_1} + e^{-(\sigma+\tau)\tilde{v}_1}} \Big\{ e^{(\sigma-\tau)/\tilde{v}_1} \sinh\sigma \\ &\quad + e^{-(\sigma+\tau)\tilde{v}_1} \frac{(1-i\tilde{v}_1)^2((1+\tilde{v}_1^2)\sinh\sigma + 2\tilde{v}_1 \cosh\sigma)}{1-\tilde{v}_1^4} \Big\}. \end{aligned}$$

Energy of the spike:

$$E^1_{\text{spike}}(\epsilon_1) \equiv E - S - E_0 = \frac{\sqrt{\lambda}}{2\pi}\left[\frac{1}{2}\ln\frac{1+\epsilon_1^{-1}}{1-\epsilon_1^{-1}} - \epsilon_1^{-1}\right]$$

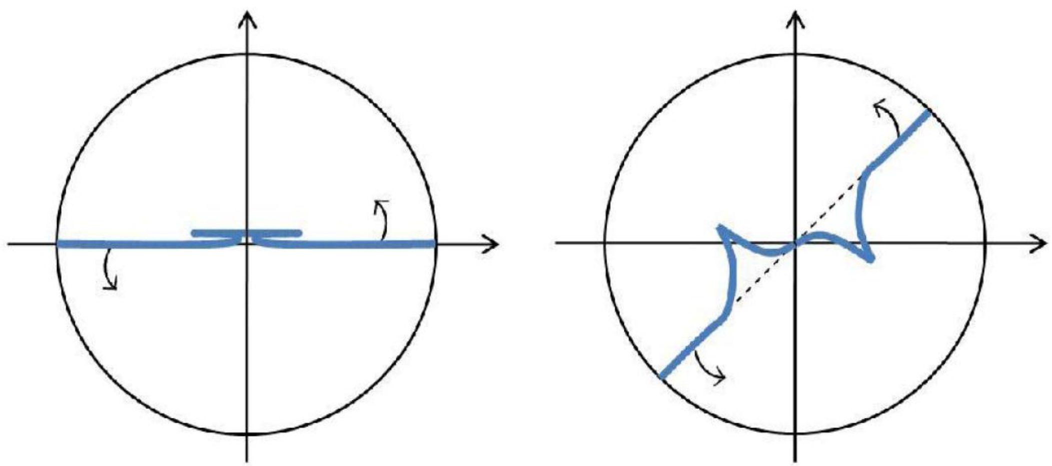
N. Dorey, M. Losi: 1001. 4750; AJ, K. Jin: 1001.5301

Two-spike dynamics:

$$E - S = \frac{\sqrt{\lambda}}{2\pi}\left[\ln\frac{8\pi S}{\sqrt{\lambda}} + \frac{1}{2}\ln\frac{1+\epsilon_1^{-1}}{1-\epsilon_1^{-1}} + \frac{1}{2}\ln\frac{1+\epsilon_2^{-1}}{1-\epsilon_2^{-1}} - \epsilon_1^{-1} - \epsilon_2^{-1}\right]$$

The energy of the two-spike solution is the sum of two individual spike energies:

$$E^2_{\text{spike}}(\epsilon_1,\epsilon_2) = E^1_{\text{spike}}(\epsilon_1) + E^1_{\text{spike}}(\epsilon_2)$$



(a) $t = 0$

(b) $t = \frac{\pi}{4}$

Euclidean world-sheet:

- Relevant for the Alday-Maldacena program
- Example of one-soliton

Alday, Maldacena: 0705.0303

$$\alpha(z,\bar{z}) = \frac{1}{2}\ln\left[\frac{1}{2}\tanh^2\frac{\tau}{\sqrt{2}}\right]$$

- One regularized cusp near the boundary of AdS_3 at $r=r_c$:

$$Y_{a\dot{a}} = \begin{pmatrix} \frac{1}{2e^\tau r_c}\left(2+\sqrt{2}\tanh\frac{\tau}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}e^\sigma\tanh\frac{\tau}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}e^{-\sigma}\tanh\frac{\tau}{\sqrt{2}} & -\frac{e^\tau r_c}{2}\left(-2+\sqrt{2}\tanh\frac{\tau}{\sqrt{2}}\right) \end{pmatrix}$$

- Originally constructed by Berkovits and Maldacena.

Berkovits, Maldacena: 0807.3196

N-spike solution:

Sinh-Gordon : $\hat{\alpha}(z', \bar{z}') = \ln\Big[\frac{4\zeta}{i}\frac{\partial(\varphi_1+\varphi_2)}{\varphi_1-\varphi_2}\Big]$

$$\varphi_1(\zeta, z', \bar{z}') = -\Big(\sum_{j,l=1}^{N}\frac{\lambda_j}{\zeta+\zeta_j}(1-A)^{-1}_{jl}\lambda_l\Big)e^{i\zeta\bar{z}'-iz'/4\zeta},$$

$$\varphi_2(\zeta, z', \bar{z}') = \Big(1+\sum_{j,l,k=1}^{N}\frac{\lambda_j}{\zeta+\zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j+\zeta_l}(1-A)^{-1}_{lk}\lambda_k\Big)e^{i\zeta\bar{z}'-iz'/4\zeta},$$

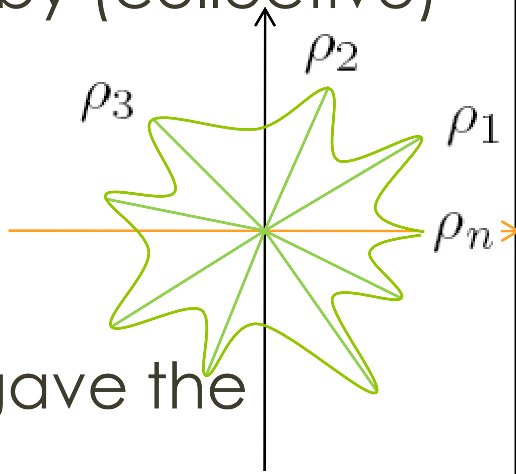
Spiky Strings : AJ, K. Jin: 0903.3389,1001.5301

$$\begin{aligned} Z_1 &= \frac{1-i}{4}e^{\frac{i}{2}(\bar{z}'-z')}\Big\{i(\tilde{\varphi}_2-\tilde{\varphi}_1)^1_+\Big[e^{-\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2+\tilde{\varphi}_1)^2_+ - ie^{\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2+\tilde{\varphi}_1)^2_-\Big] \\ &+(\tilde{\varphi}_2+\tilde{\varphi}_1)^1_+\Big[e^{-\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2-\tilde{\varphi}_1)^2_+ + ie^{\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2-\tilde{\varphi}_1)^2_-\Big]\Big\}, \\ Z_2 &= \frac{1+i}{4}e^{\frac{i}{2}(\bar{z}'-z')}\Big\{i(\tilde{\varphi}_2-\tilde{\varphi}_1)^1_+\Big[e^{-\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2+\tilde{\varphi}_1)^2_+ + ie^{\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2+\tilde{\varphi}_1)^2_-\Big] \\ &+(\tilde{\varphi}_2+\tilde{\varphi}_1)^1_+\Big[e^{-\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2-\tilde{\varphi}_1)^2_+ - ie^{\frac{1}{2}(\bar{z}'+z')}(\tilde{\varphi}_2-\tilde{\varphi}_1)^2_-\Big]\Big\}. \end{aligned}$$

Moduli dynamics:

- Spike locations can be described by (collective) coordinates: $\rho_i(t)$, $i = 1,2,\cdots,n$ with an interacting Lagrangian $L(\rho,\dot{\rho})$

$$Z_1^i(\tau) = Z_1(\tau, \sigma_i(\tau)), \quad Z_2^i(\tau) = Z_2(\tau, \sigma_i(\tau)),$$



- In the near static limit, Kruczenski gave the interaction potential:

$$V = \sum_j \ln\left\{\sin^2 \frac{\theta_{j+1} - \theta_j}{2}\right\}$$

Kruczenski '04
Freyhult,Kruczenski,Tirziu '09

- An exact description should be deduced from the dynamical n-spike solution: it is related to the system describing n-solitons.

⇒ Dynamical system of Calogero (RS) type

III. R×S²: Giant magnons

- Time-like conformal gauge:

$$t = X^0 = \tau \qquad (\partial_\pm \vec{X})^2 = 1$$

- Equation of motion:

$$\partial_+ \partial_- \vec{X} + (\partial \vec{X})^2 X = 0$$

- Reduction:

$$\cos\alpha = \partial_+ \vec{X} \cdot \partial_- \vec{X}$$

$$\partial^2 \alpha - \sin\alpha = 0 \qquad \alpha = 4\tan^{-1}[e^{-\gamma(\sigma - v\tau)}]$$

- One-magnon solution:

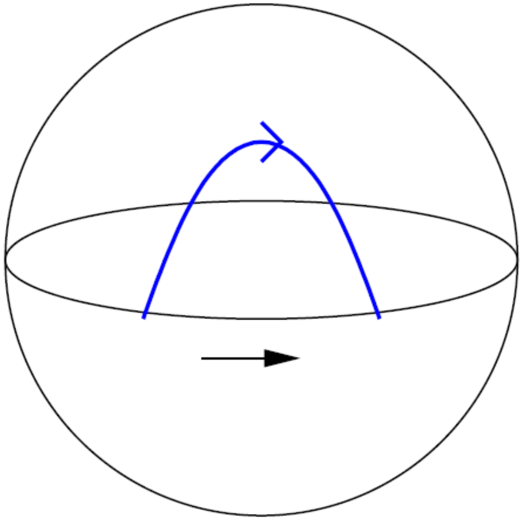
$$Z_1 = X_1 + iX_2 = e^{i\tau}(v + i\sqrt{1-v^2}\tanh[\gamma(\sigma - v\tau)])$$

$$X_3 = \sqrt{1-v^2}\,\mathrm{sech}[\gamma(\sigma - v\tau)]$$

Energy of Giant magnon:

- Energy of the soliton:

$$\epsilon_{\rm sol} = \gamma = \frac{1}{\sqrt{1-v^2}}$$



- Energy of Giant magnon is inversely proportional to the energy of the soliton:

$$E_{\rm mag} = \sqrt{1-v^2} = \frac{1}{\epsilon_{\rm sol}}$$

Hofman, Maldacena: hep-th/**0604135**

- Two magnons:

$$E^2_{\rm mag} = \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}$$

Scattering of Giant magnons

- Scattering of solitons:

$$\Delta t = \frac{2}{\sinh\theta}\ln[\tanh\theta]$$

- Same time delay:

$$t = \tau \quad \Rightarrow \quad \frac{\partial \delta_{\rm sol}}{\partial \epsilon_{\rm sol}} = \Delta\tau_{\rm sol} = \Delta t_{\rm mag} = \frac{\partial \Delta_{\rm mag}}{\partial E_{\rm mag}}$$

- Different phase shift:

$$\epsilon_{\rm sol} \neq E_{\rm mag} \quad \Rightarrow \quad \delta_{\rm sol} \neq \Delta_{\rm mag}$$

- Different dynamical system

I. Aniceto, AJ: 0810.4548

Puzzle

- The moduli (collective coordinates) of sine-Gordon solitons and the stringy magnons obey the same equations of motion:

Time is the same!

$$\{X_i(t)\} \leftrightarrow \{\rho_i(t)\}$$

- From the energy we deduce

$$H_{\rm sol} = \mathrm{Tr}(\hat{L}) \neq H_{\rm mag} \overset{?}{=} \mathrm{Tr}(\hat{L}^{-1})$$

- But seemingly this is in contradiction with the fact that the time is the same

⇨ Different Lagrangians --- Same EOM.

Non-local Lagrangian for sine-Gordon:

- The sine-Gordon equation $\partial^2\phi - \sin\phi = 0$ is associated with the local Lagrangian:

$$\mathcal{L}_{\mathrm{sG}} = \frac{1}{2}(\partial\phi)^2 - \cos\phi$$

- But the stringy ($R\times S^2$) reduced Lagrangian differs from $\mathcal{L}_{\mathrm{sG}}$
- Also true for the full quantum theory
- One can perform the Pohlmeyer reduction at the level of path integral:

$$Z = \int dJ\delta(F(J))e^{i\int d^2x \mathrm{Tr}(J^2)}$$

where $F_{\mu\nu} = \partial_\mu J - \partial_\nu J + [J_\mu, J_\nu]$

for a SU(N) chiral model.

B.E. Fridling, AJ: '84 ; Tseytlin, Fradkin: '84

S^2=SO(3)/SO(2) coset model:

- The Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\partial_\mu g^{-1}\partial^\mu g) = -\frac{1}{2}\mathrm{Tr}(J_\mu J^\mu)$$

where the currents are

$$J_\mu = g^{-1}\partial_\mu g = \sum_i t^i J^i_\mu$$

- The third component will be gauged away, while the first two remain dynamical.

$$J^3_\mu \equiv A_\mu \qquad \Pi = J^1 + iJ^2$$

- The Bianchi identity: $\left(\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu]\right)^i = 0$

$$\partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{2}(\bar{\Pi}_\mu \Pi_\nu - \bar{\Pi}_\nu \Pi_\mu) = 0$$

$$\epsilon^{\mu\nu} D_\mu J^a_\nu = \epsilon^{\mu\nu}\left(\partial_\mu J^a_\nu - i\epsilon_{ab}A_\mu J^b_\nu\right) = 0$$

Constraints

The non-local Lagrangian:

- Introduce the light-cone notation:

$$\Pi_\pm = \Pi_0 \pm \Pi_1, \quad A_\pm = A_0 \pm A_1, \quad \partial_\pm = \partial_0 \pm \partial_1,$$

- The Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}(\bar{\Pi}_-\Pi_+ + \bar{\Pi}_+\Pi_-) + \frac{\nu}{2}\Big(\partial_-A_+ - \partial_+A_- + \frac{1}{2}(\bar{\Pi}_-\Pi_+ - \bar{\Pi}_+\Pi_-)\Big) \\ & + \frac{1}{2}\lambda(D_-\bar{\Pi}_+ - D_+\bar{\Pi}_-) + \frac{1}{2}\bar{\lambda}(D_-\Pi_+ - D_+\Pi_-).\end{aligned}$$

- $A_+, \Pi_+, \bar{\Pi}_+$ are Lagrange multipliers
- The non-local Lagrangian

$$\mathcal{L} = -\frac{\nu}{2}\partial_+A_- - \frac{1}{2}\lambda\partial_+\bar{\Pi}_- - \frac{1}{2}\bar{\lambda}\partial_+\Pi_- = -\partial_-\nu\partial_+\phi$$

where $\nu = \left(\partial_- q_-^{-1}(\partial_-^2 - 1) + 4q_-\partial_-\right)^{-1}2$

Poisson structures:

- Usual Poisson bracket: $\{\phi(x),\phi(x')\} = \delta'(x-x')$

$\theta_0 = \partial_-$

- Integrable theories have a sequence of compatible Poisson brackets.
- For the non-local Lagrangian:

$$\mathcal{L} = F(\phi)\partial_+\phi \Rightarrow \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = F(\phi)$$

- Non-local Poisson bracket

$$\{\pi,\phi\} = \delta \qquad \{\phi(x),\phi(x')\} = \theta_m^{-1}\delta(x-x')$$

$$\theta_m = (\theta_1+\theta_0)\theta_1^{-1}(\theta_1+\theta_0) \qquad \theta_1 = \partial_-^3 + 4\partial_- q_- \partial_-^{-1} q_- \partial_-$$

- Derived originally by A. Mikhailov: hep-th/**0511069**
- Quantum theory measure:

$$\int d\phi\, |\{F(\phi),F(\phi)\}|^{\frac{1}{2}}\, e^{i\int d^2x \mathcal{L}_{\text{non-local}}}$$

AJ, K. Jin: in preparation

Dynamical system: spikes vs solitons

- For soliton-soliton scattering:

$$\phi_{ss} = \ln\left[\frac{v\cosh(\gamma x) - \cosh(\gamma v t)}{v\cosh(\gamma x) + \cosh(\gamma v t)}\right]^2,$$

- Follow the poles of the Hamiltonian density, we find the trajectories of the poles: G. Bowtell and A.E.G. Stuart, '77

$$x(t) = \pm\frac{1}{\gamma}\cosh^{-1}\left[\frac{1}{v}\cosh(\gamma v t)\right]$$

- The N-body Hamiltonian: S.N.M. Ruijsenaars and H. Schneider, '86

$$H = \sum_{j=1}^{N}\cosh\theta_j \prod_{k\neq j} f(q_j - q_k),$$

- Soliton-soliton scattering potential:

$$W_r(q) = \left|\coth\left(\frac{q}{2}\right)\right|$$

- Integrable, Lax matrix L: $H = \mathrm{Tr}(L + L^{-1}) \equiv \mathrm{Tr}(\hat{L})$

Moduli space dynamics: N-body Hamiltonian

- N-solitons:

$$h_s = \mathrm{Tr}(\hat{L}) = \sum_{i=1}^{N} \cosh\theta_i \prod_{j\neq k} \tanh|q_j - q_k|$$

- N-magnons:

$$H_m = \mathrm{Tr}(\hat{L}^{-1}) \Rightarrow E_m = \sum_i \frac{1}{\epsilon_i}$$

- How are the equations of motion to be the same?

Different Poisson structure:

- Lagrangians:

$$L = p_i \; w(p,q)_{ij} \; \dot{q}_j - H(p,q)$$

we have

$$L_{\text{sol}} = \sum \theta_i \dot{q}_i - h_{\text{sol}}(\theta, q) \;\longleftrightarrow\; \{q_i, \theta_j\} = \delta_{ij}$$

$$L_{\text{mag}} = \theta \cdot w \cdot \dot{q} - H_{\text{mag}}(\theta, q) \;\longleftrightarrow\; \{q, \theta\} = (w^{-1})_{ij}$$

- Theorem: Integrable systems not only have a sequence of Hamiltonians,

$$H_n = \text{Tr}(\hat{L}^n) \;\longleftrightarrow\; \Omega_n$$

but also a sequence of Poisson structures!

Equations of motion:

- Hamilton's equations

$$\dot{z} = \Omega_m \nabla_z H_n = \Omega_{m-k} \nabla_z H_{n+k}$$

- The string and soliton use different Hamiltonian and different Poisson structures to keep the same equations of motion.

Tractable dynamical model:

- Separated magnons: $q_{i-1} \ll q_i$

$$H = \sum e^{\theta_i} V_i(q)$$

$$V_i(q) = f(q_{i-1} - q_i) f(q_i - q_{i+1})$$

- Symplectic form:

$$w = dq_i \wedge d\theta_i \qquad \{q_i, \theta_i\} = \delta_{ij}$$

- Conserved quantities:

$$h_n = \mathrm{Tr}(L^n)$$

- Hierarchy:

$$\pi_j \bigtriangledown h_k = \pi_{j-l} \bigtriangledown h_{k+l}$$

Poisson structure for magnons:

- The soliton Hamiltonian

$$h_s = h_1 + h_{-1}$$

- For magnons:

$$H_m = \mathrm{Tr}\left(\frac{1}{L + L^{-1}}\right) = \sum_{n=0}^{\infty} (-1)^n h^{2n+1}$$

- Equations of motion coincide:

$$\pi_m \bigtriangledown H_m = \pi_2 \bigtriangledown h_s$$

- With the Poisson bracket:

$$\pi_m = \pi_0 + 2\pi_2 + \pi_4$$

I. Aniceto, AJ: 0810.4548; I. Aniceto, J. Avan, AJ: 0912.3468

Conclusions:

- We used the inverse scattering technique to construct a (most) general set of AdS string classical solutions;
- This construction features a one-to-one correspondence between spikes (of the string) and solitons;
- Discussed the dynamics of spikes/magnons and their moduli space.

Future:

HAPPY BIRTHDAY YOICHI