(Discretized) Minimal Surface in AdS

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S. Dobashi, KI and K. Iwasaki, arXiv:0805.3594, JHEP 07 (2008)088
S. Dobashi and KI, arXiv:0901.3046, Nucl. Phys. B819 (2009) 18
KI, K. Iwasaki, work in progress
Y. Hatsuda, KI, K. Sakai, Y. Satoh, work in progress

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On the occasion of Prof. Kazama's 60th birthday

Outline

1. Introduction
2. Discretized Minimal Surface in AdS
3. Thermodynamic Bethe Ansatz and Remainder function
4. Outlook

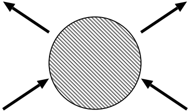
Introduction

AdS/CFT Correspondence

$N = 4$ $U(N)$ SYM $\Longleftrightarrow$ type IIB Superstrings on $AdS_5 \times S^5$

Gluon Scattering Amplitudes [Alday-Maldacena]

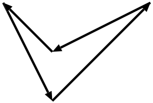
gluon scattering



weak(strong)

$\Longleftrightarrow$ T-dual

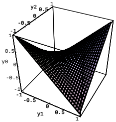
Wilson loop



weak(strong)

$\Longleftrightarrow$ AdS/CFT

Minimal surface in AdS



strong(weak)

Gluon Scattering Amplitudes in $\mathcal{N}=4$ SYM

the BDS conjecture Bern-Dixon-Smirnov, Anastasiou-Bern-Dixon-Kosower

Planar L-loop, n-point amplitude (recursive structure)

$$A_n^{(L)}(k_1,\cdots,k_n)=A_n^{(0)}(k_1,\cdots,k_n)\mathcal{M}_n^{(L)}(\epsilon)$$

$\mathcal{M}_n(\epsilon)\equiv 1+\sum_{L=1}^{\infty}\lambda^L\mathcal{M}_n^{(L)}(\epsilon)$

IR divergence: Dimensional regularization ($D=4-2\epsilon$)

$$\begin{aligned}\ln\mathcal{M}_n(\epsilon) &= \frac{A_2}{\epsilon^2}+\frac{A_1}{\epsilon}\\ &-\frac{1}{16}f(\lambda)\sum_{i=1}^{n}\left(\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)^2-\frac{g(\lambda)}{4}\sum_{i=1}^{n}\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)+\frac{f(\lambda)}{4}F_n^{(BDS)}+C\end{aligned}$$

$f(\lambda)=4\sum_{l=1}^{\infty}f_0^{(l)}\lambda^l$: cusp anomalous dimension

$g(\lambda)=2\sum_{l=1}^{\infty}\frac{f_1^{(l)}}{l}\lambda^l$: collinear anomalous dimension

For $n=4$

$$F_4^{BDS}=\frac{1}{2}\log^2\left(\frac{s}{t}\right)+\frac{2\pi^2}{3}$$

Test of the BDS conjecture (Weak Coupling)

- loop calculations ($n \leq 5$) BDS, ...
- Dual conformal invariance: determines $n \leq 5$ amplitudes
- Gluon amplitude/Wilson loop duality at weak coupling
  Drummond-Henn-Korchemsky-Sokatchev
- Discrepancy in 6-point 2-loop amplitude
  Bern-Dixon-Kosower-Roiban-Spradlin-Volovich
  Drummond-Henn-Korchemsky-Sokatchev

$$\ln M_6^{MHV} = \ln W(C_6) + const., \quad F_6^{WL} = F_6^{BDS} + R_6(u), \quad R_6 \neq 0$$

- remainder function $R_n(u)$: $F_n = F_n^{BDS} + R_n$
  Non-trivial dependence on the cross-ratios $(3n - 15)$

$$u_{ij,kl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \quad (x_{ij}^2 = t_i^{[j-i]})$$

For $n = 6$, $u_{13,46}$, $u_{24,15}$, $u_{35,26}$ are independent cross ratios.

Minimal Surface in AdS

AdS_5: embedding coordinates in $\mathbf{R}^{2,4}$: isometry $SO(2,4)$

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

Poincaré coordinates: (y^μ, r) $\mu = 0,1,2,3$

$$Y^\mu = \frac{y^\mu}{r}, \quad Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

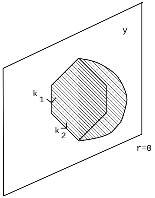
Action in confomal gauge:

$$S = \frac{R^2}{2\pi} \int d^2 z \frac{\partial y^\mu \bar{\partial} y_\mu + \partial r \bar{\partial} r}{r^2}$$

Euler-Lagrange equation+boundary condition

- ends at $r \to 0$ (AdS boundary)
- y^μ: light-like segments

$$\mathcal{M}_n \sim \exp(-S[y,r])$$



$S[y,r]$ = Area of the minimal surface (Plateau problem)

Alday-Maldacena's Solution (4pt amplitude)

Alday-Maldacena arXiv:0705.0303

- AdS_3 constraint: $Y_3 = Y_4 = 0$

$$r^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 = 1$$

- static gauge: $r = r(y^1, y^2)$, $y^0 = y^0(y^1, y^2)$
- dimesnional regularization $D = 3 - 2\epsilon$
- conformal boost: $s = t \to$ general (s, t)

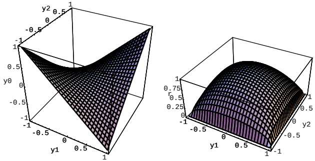
4-point amplitude $(s = t)$: $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$

boundary condition:

$r(\pm 1, y_2) = r(y_1, \pm 1) = 0$,

$y_0(\pm 1, y_2) = \pm y_2$, $y_0(y_1, \pm 1) = \pm y_1$

$y_0 = y_1 y_2$, $r = \sqrt{(1 - y_1^2)(1 - y_2^2)}$



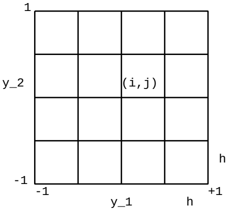
$$S_4 = \ln \mathcal{M}_4 \quad f(\lambda) \sim \sqrt{\lambda}$$

minimal surface (2)

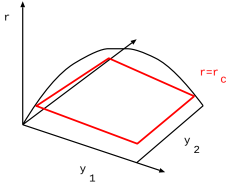
- Euler-Lagrange eqs. with AdS_d constraint (de Vega Sanchez)
  Static gauge, Pohlmeyer reduction
  Sinh-Gordon eq. (AdS_3) Jevicki-Jin-Kalousios-Volovich
  B_2 Toda (AdS_4) Burrington-Gao, Alday-Gaiotto-Maldacena
  Hitchin equation $G = SU(4)$ with Z_4 automorphism
  Alday-Gaiotto-Maldacena
- divergent at cusps
  - dimesnional regularization ($D = 3 - 2\epsilon$)
  - radial cutoff regularization ($r \geq r_c$)
- Solutions:
  - Sinh-Gordon eq. 8-pt (AdS_3) Alday-Maldacena
    Hitchin eq. 6-pt (AdS_5) Alday-Gaiotto-Maldacena
  - power series solutionItoyama-Mironov-Morozov, Jevicki-Jin
  - Numerical SolutionDobashi-KI-Iwasaki

Discretized Minimal Surfaces in AdS

- discretization
  - square lattice with spacing $h = \frac{2}{M}$
  - (i,j) $(i,j = 0,\cdots,M)$
    $y_0[i,j] = y_0(-1+hi, -1+hj)$
    $r[i,j] = r(-1+hi, -1+hj)$.



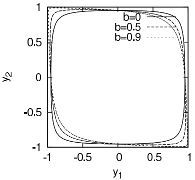
- $2(M-1)^2$ nonlinear simultaneous equations for $y_0[i,j]$ and $r[i,j]$
- action $S^{dis}[r_c] = \sum_{r[i,j]\geq r_c} L[i,j]h^2$
  radial cut-off regularization
  - $M = 520$ (Dobashi-KI-Iwasaki, Dobashi-KI)
  - AdS_3 constraint: single eom for y_0
    $M = 1200$



Minimal surface:4-point amplitude (Exact solution)

(s,t) solution: $-(2\pi)^2 s = -8a^2/(1-b)^2$, $-(2\pi)^2 t = -8a^2/(1+b)^2$

$$S_4[r_c, b] \;=\; \int_S dy_1 dy_2 L, \quad L = \frac{1}{(1-y_1^2)(1-y_2^2)}$$



S: region surrounded by the cut-off curve C

$$r_c^2 = (1-y_1^2)(1-y_2^2)\frac{1}{(1+by_1y_2)^2}$$

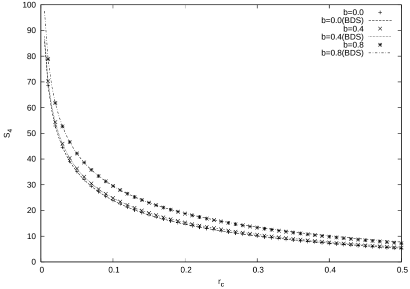
Expand around $r_c = 0$

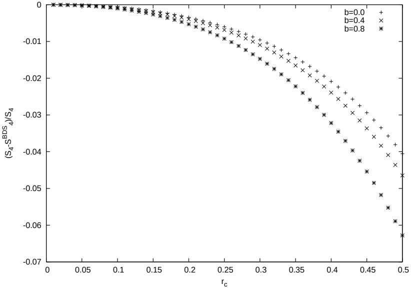
$$S_4[r_c, b] = \frac{1}{4}\log^2\left(\frac{r_c^2}{-8\pi^2 s}\right) + \frac{1}{4}\log^2\left(\frac{r_c^2}{-8\pi^2 t}\right) - \frac{1}{4}\log^2(\frac{s}{t})$$

$$+a_0 + a_1 r_c^2 \log(r_c^2) + a_3 r_c^2 + O(r_c^4 \log r_c^2) \qquad a_0 = -3.289...$$

$$F_4^{BDS} = -\frac{1}{4}\log^2(\frac{s}{t}) - \frac{\pi^2}{3} = -\frac{1}{4}\log^2(\frac{s}{t}) - 3.28987...$$

Exact Formula vs the BDS Formula (finite r_c correction)





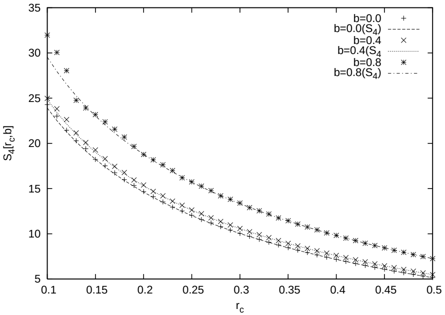
$S_4[r_c, b]$ and $S_4^{BDS}[r_c, b]$ $\qquad$ $(S_4 - S_4^{BDS})/S_4$

The BDS formula (b:conformal boost parameter)

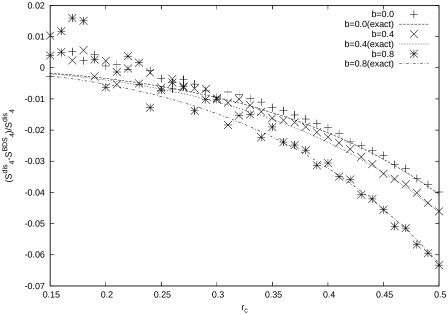
$$S_4^{BDS}[r_c, b] = \frac{1}{4}\log^2\left(\frac{r_c^2(1-b)^2}{16}\right) + \frac{1}{4}\log^2\left(\frac{r_c^2(1+b)^2}{16}\right) - \frac{1}{4}\left\{\log\left(\frac{1+b}{1-b}\right)^2\right\}^2 - \frac{\pi^2}{3}$$

Numerical check of the BDS formula: 4-pt amplitude

$M = 520$ Dobashi-KI



$S[r_c, b]$ vs $S^{dis}[r_c, b]$



$(S_4^{dis} - S_4^{BDS})/S_4^{dis}$

- $(S_4^{dis} - S_4)/S_4^{dis} \sim 0.1\%$ ($b = 0.4$, $r_c = 0.3$)
- Finite r_c correction $\leq 6\%$
- numerical error becomes large $r_c \leq 0.2$ (and large b)

n-point amplitude

$$\tilde{S}_n[r_c] = \frac{1}{8}\sum_{i=1}^{n}\left(\log\frac{r_c^2}{-8\pi^2 s_{i,i+1}}\right)^2 + F_n + a_0 + a_1 r_c^2 \log r_c^2 + a_2 r_c^2 + O(r_c^4 \log r_c^2)$$

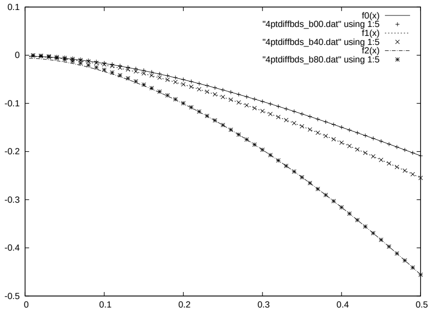
$$F_n = -\frac{1}{2}F_n^{BDS} + R_n(u_{ij,kl})$$

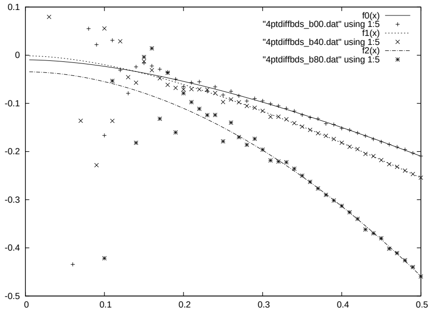
- remainder function: R_n
- finite r_c correction $\sim a_0 + a_1 r_c^2 \log r_c^2 + a_2 r_c^2$

4pt amplitudes (2)

r_c correction KI, K. Iwasaki, work in progress

$S_4[r_c, b] - S_4^{BDS}[r_c, b] \sim a_0 + a_1 r_c^2 \log(r_c^2) + a_2 r_c^2$





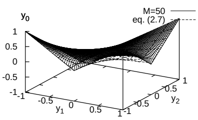
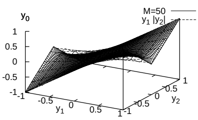
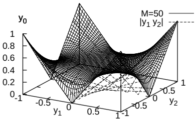
$S_4^{exact}[r_c, b] - S_4^{BDS}[r_c, b]$

$a_0 \sim -0.0014$

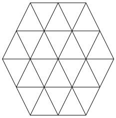
$S_4^{dis} - S_4^{BDS}$ $(M = 1200)$

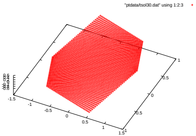
$a_0 \sim -0.01$

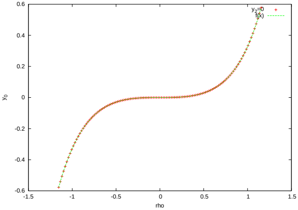
higher-point amplitudes

6-point solution1	6-point solution2	8-point
		
$(u_1 = u_2 = u_3 = 1)$	$(u_1 = u_2 = u_3 = 1)$	$u_{ij,kl}$:constant
$a_0 = -1.25$	$a_0 = -1.27$	$a_0 = 1.15$

- rectangular solution (nontrivial u_1, u_2, u_3)
- regular hexagon, octagon







$y_2 = 0$

Thermodynamic Bethe Ansatz and the Remainder Function

embedding coordinates$Y$: ($Y^2 = -1$)
action:

$$S = \int d^2z \left\{ \partial Y \bar{\partial} Y + \lambda(Y^2 + 1) \right\}$$

EOM+Virasoro Constraints

$$\partial\bar{\partial}Y + (\partial Y \bar{\partial} Y) Y = 0, \quad (\partial Y)^2 = (\bar{\partial} Y)^2 = 0$$

orthonormal basis $q = (Y, B_1, B_2, B_3, e^{-\alpha/2}\partial Y, e^{-\alpha/2}\bar{\partial} Y)$ $\quad e^{\alpha} \equiv \partial Y \bar{\partial} Y$
EOM$\Longleftrightarrow$ $SU(4)$ Hitchin eqs. with $\mathbf{Z}_4$-symm (Alday-Gaiotto-Maldacena)

$$[D_z, D_{\bar{z}}] + [\Phi_z, \Phi_{\bar{z}}] = 0, \quad D_z \Phi_{\bar{z}} = 0, \quad D_{\bar{z}} \Phi_z = 0$$

$$\Phi_z = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} e^{-\alpha/2} v_I \sigma^I \\ \frac{1}{\sqrt{2}} e^{\alpha/2} & 0 \end{pmatrix}, \quad A_z = \frac{1}{4} \begin{pmatrix} -\partial\alpha + d_{IJ}\sigma^{IJ} & 0 \\ 0 & \partial\alpha + d_{IJ}\sigma^{IJ} \end{pmatrix}$$

$v_I = B_I \partial^2 Y$, $d_{IJ} = \partial B_I B_J$

Hitchin eq and TBA

- boundary condition ($z \to \infty$)

$$A_z \sim \frac{m}{z} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \Phi \sim \frac{1}{\sqrt{2}} \begin{pmatrix} P(z)^{1/4} & 0 & 0 & 0 \\ 0 & -iP(z)^{1/4} & 0 & 0 \\ 0 & 0 & -P(z)^{1/4} & 0 \\ 0 & 0 & 0 & iP(z)^{1/4} \end{pmatrix}$$

$P(z) = \partial^2 Y \bar{\partial}^2 Y = z^{n-4} + \cdots$ for n sided polygon: $w = \int P^{1/4}(z) dz$

- linear problem: $(D_z + \zeta^{-1}\Phi_z)\psi = 0$, $(\bar{D}_{\bar{z}} + \zeta\Phi_{\bar{z}})\psi = 0$ (ζ:spectral parameter)
- Asymptotic behavior of ψ shows the Stokes phenomena
- cross ratio are given by the Stokes data b_i, monodromy $\mu = e^{i\phi}$
- $\mathcal{X}_a[\zeta]$: function of $b_i[\zeta]$ whose asymptotic behavior is uniform ($\mathcal{X}_a \sim \exp(Z_a/\zeta)$ $\zeta \to 0$) Gaiotto-Moore-Neitzke
  - Daurboux coodrinates of moduli space of N=2 theories on $R^3 \times S^1$
  - WKB-analysis: $\mathcal{X}_a \sim \exp(\frac{1}{\zeta} \oint P^{\frac{1}{4}} dz)$
  - $\mathcal{X}_a$ has dicontinuities along some rays and obeys TBA-like equations

remainder function: 6-point amplitudes

- $P(z) = z^2 - m$, parameters $Z = |Z|e^{i\psi}$, $\mu = e^{i\phi}$
- $e^{\epsilon(\theta)} = \mathcal{X}_1[e^\theta]$, $e^{\tilde{\epsilon}(\theta)} = \mathcal{X}_1\mathcal{X}_3[e^{\frac{i\pi}{4}}e^\theta]$ obey the TBA eqs of the A_3 integrable theory
- cross ratios u_1, u_2, u_3: functionals of $\epsilon(\theta)$ and $\tilde{\epsilon}(\theta)$

Remainder function

$$R_6 = \sum_{i=1}^{3}\left(\frac{1}{8}\log^2 u_i + \frac{1}{4}\mathrm{Li}_2(1-u_i)\right) - |Z|^2 - A_{free} + \mathrm{const.}$$

Free energy of the A_3 system

$$A_{free} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\theta 2|Z|\cosh\theta\log(1+e^{-\epsilon}\mu)(1+\frac{e^{-\epsilon}}{\mu})+2\sqrt{2}|Z|\cosh\theta\log(1+e^{-\tilde{\epsilon}})$$

$u_1 = u_2 = u_3 = u$ (conformal point, μ: chemical potential)

$$R_6(u,u,u) = -\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 + \frac{3}{8}(\log^2 u + 2\mathrm{Li}_2(1-u)), \quad u = \frac{1}{4\cos^2(\phi/3)}$$

TBA equation for A_3 theory

Klassen-Melzer mass $m_{\tilde{\epsilon}} = 2\sqrt{2}|Z|$, $m_{\epsilon_1} = m_{\epsilon_2} = 2|Z|$

$$\begin{aligned}
\epsilon(\theta) = \quad & 2|Z|\cosh\theta + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) \\
& + \frac{1}{2\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu}) \\
\tilde{\epsilon}(\theta) = \quad & 2\sqrt{2}|Z|\cosh\theta + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) \\
& + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu})
\end{aligned}$$

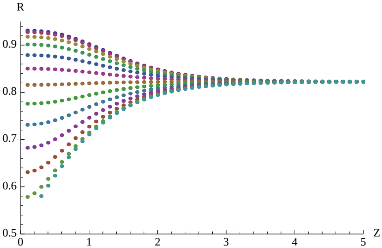
comparison with other calculations

2-loop Anastasiou-Brandhuber-Heslop-Khoze-Spence-Travaglini

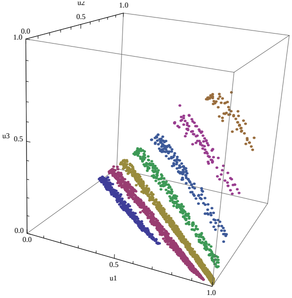
$$R_6(\frac{1}{4},\frac{1}{4},\frac{1}{4}) = 0.930933, \quad R_6^{2-loop} = 1.08916$$

$$R_6(1,1,1) = 0.523599, \quad R_6^{2-loop} = -2.706, \quad R_6^{dis} = -1.25$$

Remainder function at (u_1, u_2, u_3)
Y.Hatsuda, KI, K. Sakai, Y. Satoh, work in progress



$(|Z|, R)$ graph for various ϕ



plot of (u_1, u_2, u_3) with fixed $R = 0.9, 0.85, \cdots 0.6$

- Solve TBA equation numerically
- $R(|Z|e^{i\psi}, \mu = e^{i\phi})$, R is independent of ψ
- small Z: mass perturbation of $\mathbf{Z}_4$-parafermion $SU(2)_4/U(1)$
- large Z: $(u_1, u_2, u_3) = (1 - u_3, 0, u_3)$ colinear limit

n-point amplitudes

To determine the remainder function for n-point amplitudes, we need to

- find TBA equations
- solve them perturbatively or numerically

Minimal surface with $2n$ light-like polygonal boundary
AdS_3-constraints: $P(z) = p(z)^2$ $(p(z) = z^{n-3} + \cdots)$

- $n = 4$ TBA eq. becomes trivial Alday-Maldacena
- $n = 5$ TBA eqs. : $SU(3)_2/U(1)^2$
- $n = 6$ TBA eqs. : $SU(4)_2/U(1)^3$

general n: $SU(n-2)_2/U(1)^{n-3} \simeq (SU(2)_1)^{(n-2)}/SU(2)_{n-2}$
homogeneous Sine-Gordon model Castro-Alvaredo-Fring-Korff-Miramantos

- conformal point $p(z) = z^{n-3}$ (regular $2n$-gon) $A_{free} = \frac{\pi}{6}c$, $c = \frac{(n-2)(n-3)}{n}$
- The number of deformation parameters: $2n - 6$
- AdS_5: $SU(n-4)_4/U(1)^{n-5} = (SU(4)_1)^{n-4}/SU(4)_{n-4}$

Outlook

- Numerical solutions of Minimal surface in AdS are quantitatively in good agreement with exact solutions.
  - application to non-AdS geometry
  - calculation of remainder function (large M)
- Minimal surfaces in AdS with light-like boundary can be solved by TBA eqs.
  To determine the remainder function for n-point amplitudes, we need to
  - find TBA equations Y. Hatsuda, KI, K. Sakai, Y. Satoh
  - solve them perturbatively or numerically
- relation to Wall Crossing Formula, Seiberg-Witten, 2d CFT

Happy Birthday, Kazama-san!