# ” Penner Type Matrix Model and Seiberg-Witten Theory”

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We consider $\mathcal{N} = 2$ supersymmetric gauge theories in 4-dimensions and study the case when the theory possesses conformal invariance.

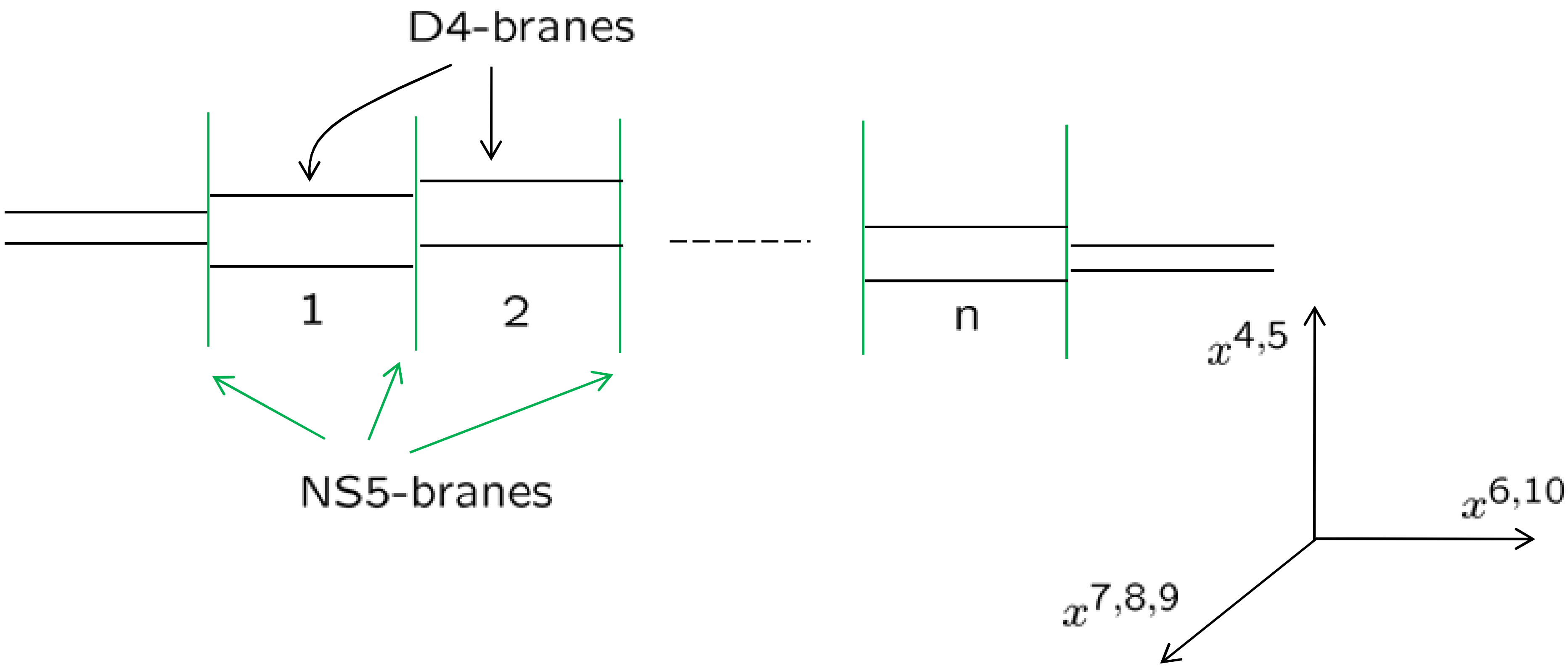
Simplest example of a conformal invariant theory:

$SU(2)$ gauge theory with $N_f = 4$ hypermultiplets

We may consider its generalizations:

Chain of $SU(2)$ gauge theories with bifundamentals and fundamental at the ends: quiver gauge theories

As is well-known, such quiver theories are obtained using the brane construction as shown in the figure:



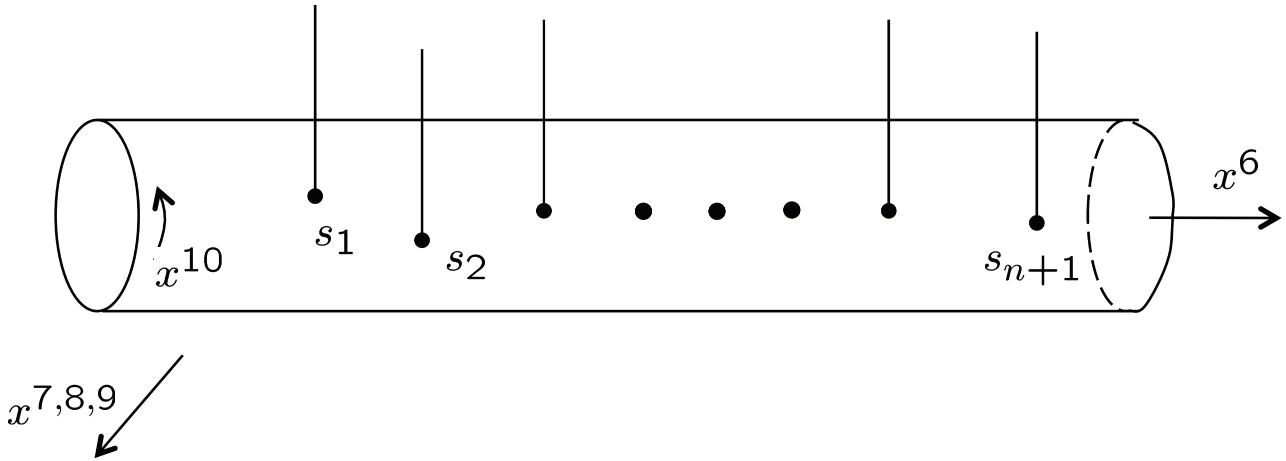

There are $n+1$ NS5 branes and a pair of $D4$ branes are suspended between neighbouring NS5 branes giving rise to $SU(2)_1 \times SU(2)_2 \cdots \times SU(2)_n$ gauge symmetry. Two D4 branes at extreme left and right extend to $x_6 = \pm\infty$ representing fundamental hypermultiplets.

Each $SU_i(2)$ theory couples to $N_f = 4$ hypermultiplets and is conformally invariant. Thus there exists a set of marginal parameters in the theory

$$\{\tau_i = \frac{\theta_i}{\pi} + \frac{8i\pi}{g_i^2}, \qquad i = 1, , n\}$$

Uplifting this brane configuration to 11 dimensions

$\Longrightarrow$ M theory picture with an M5 brane wrapping a Riemann surface (cylinder) with punctures.




Thus, conformal $\mathcal{N} = 2$ theories

$\approx$ an M5 brane wrapping a Riemann surface with a number of punctures.

Number of parameters of Riemann surface $C_{g,n}$ of genus $g$

with $n$ punctures: $3g-3+n$

This agrees with the number of gauge theory parameters $\{\tau_i\}$.

Hence one expects Gaiotto

S-duality group of quiver gauge theory $=$

mapping class group of Riemann surface $C_{g,n}$

Remarkable observation Alday,Gaiotto,Tachikawa

$$\langle \prod V_{m_i}(\tau_i) \rangle = \int [da] \; |Z_{Nek}(\tau; a; m)|^2$$

Liouville correlation function

Nekrasov partition function of $SU(2)$ gauge theory with $\epsilon_1, \epsilon_2$

Exact relationship between 4-dim CFT and 2-dim CFT.

Higher rank generalization: given by Toda theories

- direct proof ?; Shapovalov matrix, degenerate Liouville field etc

- generalities

Wilson loop, surface operators, anomaly etc.

## ♠ Matrix Model

Basic idea: Consider Liouville correlation function

$$\langle \prod_a e^{i\alpha_a \phi(q_a)} \ \prod_{i=1}^{N} \int e^{ib\phi(z_i)} \ dz_i \rangle$$

screening ops.

$$= \prod_{a<b} (q_a - q_b)^{2\alpha_a \alpha_b} \int \prod_{i,a} dz_i \, (z_i - q_a)^{2b\alpha_a} \prod_{i<j} (z_i - z_j)^{2b^2}$$

This suggests a matrix model with action Dijkgraaf,Vafa

$$S = \sum_a \alpha_a \log(M - q_a)$$

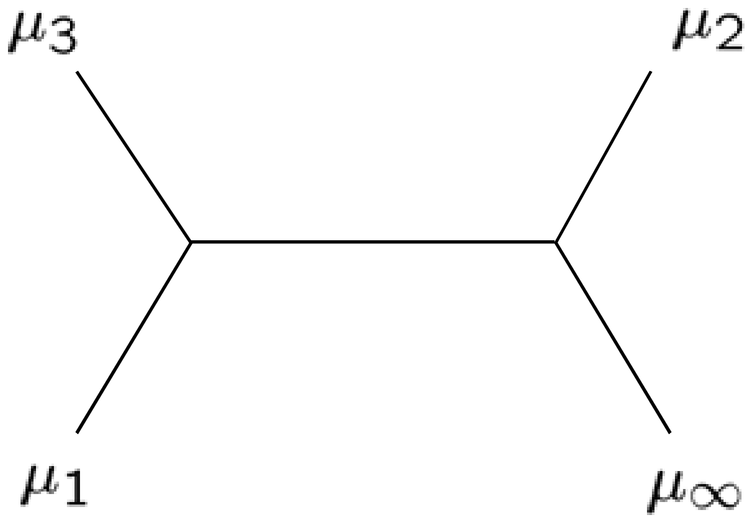
and $z_i{}'s$ are identified as matrix eigenvalues.

As we shall see that this model in fact reproduces Seiberg-

Witten theory (also for the asymptotically free cases $N_f = 2, 3$). But it still has mysterious features.

Let us consider the simple case of 4 hypermultiplets with masses $m_\pm, \tilde{m}_\pm$. Define

$$m_0 = \frac{1}{2}(m_+ - m_-),\ \ m_1 = \frac{1}{2}(\tilde{m}_+ - \tilde{m}_-)$$
$$m_2 = \frac{1}{2}(m_+ + m_-),\ \ m_3 = \frac{1}{2}(\tilde{m}_+ + \tilde{m}_-)$$




**M theory curve is given by**

$$\mathcal{C}_M: \; (v-m_+)(v-m_-)t^2 \\ +c_1(v^2+Mv-U)t+c_2(v-\tilde{m}_+)(v-\tilde{m}_-)=0$$

For convenience, set $c_1 = -(1+q),\ c_2 = q$. Then

$$\mathcal{C}_M:\ v^2(t-1)(t-q) = v(2m_2t^2 + (1+q)Mt + 2qm_3)) \\ -m_+m_-t^2 - (1+q)Ut - q\tilde{m}_+\tilde{m}_-$$

By shifting $v$ to eliminate its linear term and setting $v = xt$

$$\mathcal{C}_M : x^2 = \left(\frac{m_2t^2 + (1+q)\frac{M}{2}t + m_3q}{t(t-1)(t-q)}\right)^2 \\ + \frac{(m_0^2 - m_2^2)t^2 - (1+q)Ut + (m_1^2 - m_3^2)q}{t^2(t-1)(t-q)}$$

Seiberg-Witten differential is given by

$$\lambda_{SW} = \frac{x dt}{2\pi i} \approx \frac{m_*}{t - t_*}$$

Masses appear as residues.

Pole at $t = 0, t = \infty$; residue $\pm m_1, \pm m_0$.

Require pole at $t = 1$ with residue $\pm m_2$ and $t = q$ with residue $\pm m_3 \Longrightarrow$

$$M = \frac{-2q}{1+q}(m_2 + m_3)$$

Relation of special geometry

$$a = \int_A \lambda_{SW}, \quad a_D = \frac{\partial F}{\partial a} = \int_B \lambda_{SW}$$

## ♣ UV and IR gauge coupling constant

**Standard SW curve of $N_f = 4$ in massless case**

$$\mathcal{C}_{SW} : y^2 = 4x^3 - g_2 u x^2 - g_3 u^3$$

**Here**

$$\begin{aligned} g_2(q) &= \frac{1}{24}\left(\vartheta_3(q)^8 + \vartheta_2(q)^8 + \vartheta_4(q)^8\right), \\ g_3(q) &= \frac{1}{432}\left(\vartheta_4(q)^4 - \vartheta_2(q)^4\right) \\ &\qquad \times \left(2\vartheta_3(q)^8 + \vartheta_4(q)^4\vartheta_2(q)^4\right) \end{aligned}$$

Holomorphic differential $\omega$ is given by

$$\omega \propto \frac{dx}{y} = \frac{dx}{\sqrt{4x^3 - g_2u^2x - g_3u^3}}$$

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{4\pi}\int_A \frac{dx}{y} = \frac{1}{2\sqrt{2u}}$$

On the other hand M theory curve in the masssless limit is given by

$$\mathcal{C}_M : x^2 = -\frac{(1+q')U}{t(t-1)(t-q')}$$

Here $U$ is related to $u = tr\phi^2$ as

$$U = Au$$

and we have used $q'$ in order to distinguish it from $q$ of $\mathcal{C}_{SW}$.

SW differential is given by

$$\omega \propto \sqrt{\frac{-(1+q')A}{u}} \frac{dt}{\sqrt{t(t-1)(t-q')}}$$

After shift $t \to t + (1+q')/3$ and rescaling $t = 4z$

$$\begin{aligned} & t(t-1)(t-q') \\ & = 16\left(4z^3 - \frac{1}{12}(1-q'+q'^2)z - \frac{1}{432}(2-3q'-3q'^2+2q'^3)\right) \end{aligned}$$

By comparing with the definition of $g_2, g_3$

$$\begin{aligned}
g_2(q) &= \frac{1}{24}\left(\vartheta_3(q)^8+\vartheta_2(q)^8+\vartheta_4(q)^8\right)\\
&= \frac{1}{12}\vartheta_3(q)^8\left(1-\frac{\vartheta_2(q)^4}{\vartheta_3(q)^4}+\frac{\vartheta_2(q)^8}{\vartheta_3(q)^8}\right)\\
g_3(q) &= \frac{1}{432}\left(\vartheta_4(q)^4-\vartheta_2(q)^4\right)\left(2\vartheta_3(q)^8+\vartheta_4(q)^4\vartheta_2(q)^4\right)\\
&= \frac{1}{432}\vartheta_3(q)^{12}(2-3\frac{\vartheta_2(q)^4}{\vartheta_3(q)^4}-3\frac{\vartheta_2(q)^8}{\vartheta_3(q)^8}+2\frac{\vartheta_2(q)^{12}}{\vartheta_3(q)^{12}})
\end{aligned}$$

we notice

$$q'=\frac{\vartheta_2(q)^4}{\vartheta_3(q)^4},\qquad A=\frac{1}{\vartheta_2(q)^4+\vartheta_3(q)^4}$$

We regard $q$ in SW curve as the gauge coupling in the infrared regime $q = q_{IR}$ and $q'$ in M theory curve in the ultraviolet regime $q' = q_{UV}$.

Relation

$$q_{UV} = \frac{\vartheta_2(q_{IR})^4}{\vartheta_3(q_{IR})^4}$$

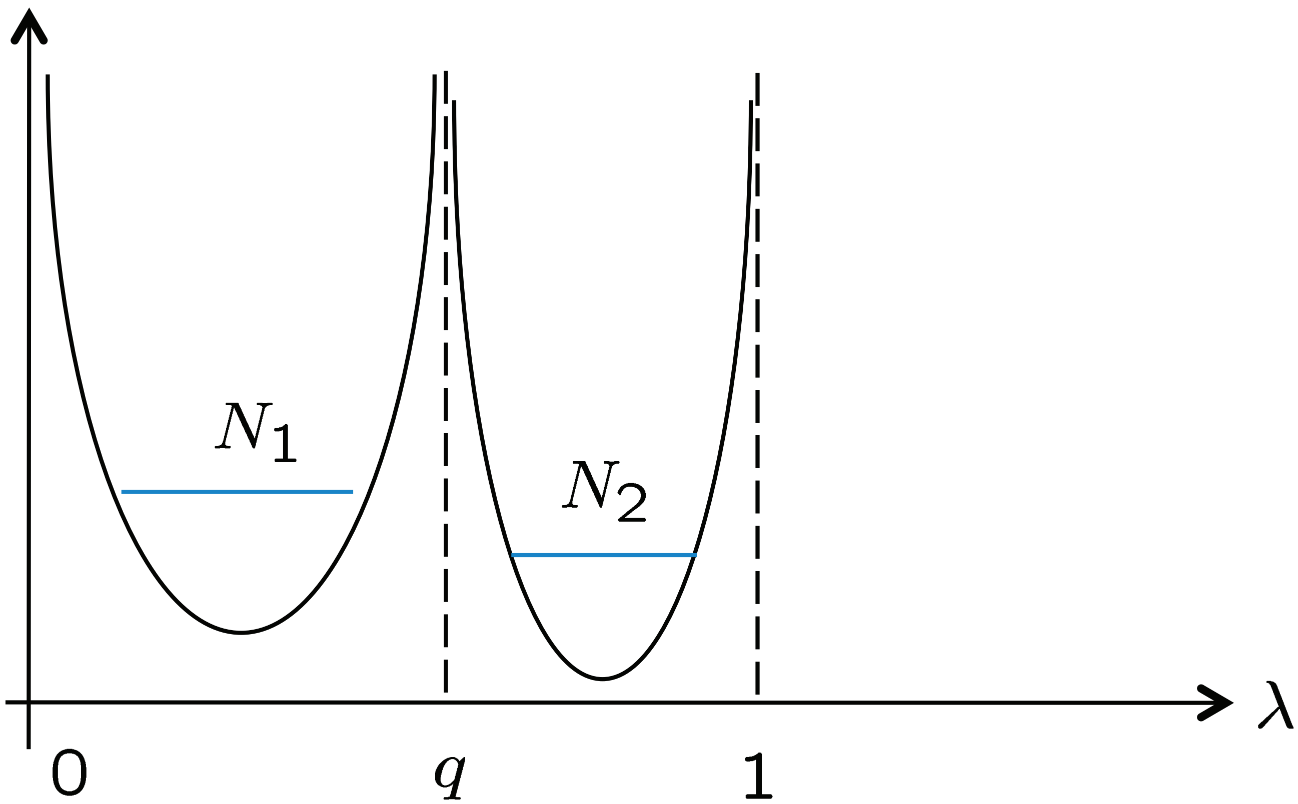
has been noticed by various authors. Grimm et al, Marshakov et al

♠ Matrix model and modular invariance

Equation of motion of matrix model is given by

$$\sum \frac{m_i}{\lambda_I - q_i} + 2g_s \sum_{I \neq J} \frac{1}{\lambda_I - \lambda_J} = 0$$

We have $q_1 = 0, q_2 = 1, q_3 = q_{UV}$. Eigenvalue distribution will look like given in the figure.



Resolvent of the theory is defined by

$$R_m(z) = g_s Tr \frac{1}{z - M}$$

which satisfies the loop equation

$$\langle R_m(z) \rangle^2 = -\langle R_m(z) \rangle W'(z) + \frac{f(z)}{4}$$
$$f(z) = 4 g_s Tr \left\langle \frac{W'(z) - W'(M)}{z - M} \right\rangle = \sum_{i=1}^{3} \frac{c_i}{z - q_i}$$

Matrix model curve (spectral curve) is defined by the discriminant of the loop equation

$$\mathcal{C}_{matrix} : x^2 = W'(z)^2 + f(z)$$
$$= \left(\frac{m_1}{z} + \frac{m_2}{z-1} + \frac{m_3}{z-q}\right)^2 + \frac{(m_0^2 - (\sum_i m_i)^2 z + qc_1}{z(z-1)(z-q)}$$

Eq. of motion $\Longrightarrow \sum_i c_i = 0$

Residue at $\infty$ being $\pm m_0 \Longrightarrow c_2 + qc_3 = m_0^2 - (\sum m_i)^2$

Then

$$qc_1 = (1+q)m_1^2 + (1-q)m_3^2 + 2qm_1m_2 - 2qm_2m_3$$
$$+2m_1m_3 - (1+q)U$$

## • Modular invariance

Consider the massless limit of spectral curve (use $q$ instead of $q_{UV}$)

$$x^2 = -\frac{(1+q)U}{z(z-1)(z-q)} = -\frac{\frac{u}{\theta_3^4}}{z(z-1)(z-q)}$$

This is invariant under

$$I : (z, x) \longrightarrow (1-z, x), \qquad q \longrightarrow 1-q, \ u \longrightarrow -u, \ S$$
$$II : (z, x) \longrightarrow (\frac{1}{z}, -z^2 x), \qquad q \longrightarrow \frac{1}{q}, \quad u \longrightarrow u, \ STS$$

Recall $q = \frac{\theta_2^4}{\theta_3^4}$.

Consider massive case. Under the S- and STS-transformations mass parameters are transformed into each other

$$I : (0,1,q,\infty) \longrightarrow (1,0,1-q,\infty), \quad m_1 \leftrightarrow m_2$$
$$II : (0,1,q,\infty) \longrightarrow (\infty,1,\frac{1}{q},0), \quad m_0 \leftrightarrow m_1$$

Under these transformations, the spectral curve should be invariant. We impose the conditions

$$x^2(z;m_0,m_1,m_2,m_3;q) = x^2(1-z;m_0,m_2,m_1,m_3;1-q)$$
$$x^2(z;m_0,m_1,m_2,m_3;q) = \frac{1}{z^4}x^2(\frac{1}{z};m_1,m_0,m_2,m_3:\frac{1}{q})$$

Requirement of modular invariance determines completely the mass dependence of the parameter $U$. Solution to the above conditions is given by

$$(1+q)U = \frac{u}{\vartheta_3^4} - q(m_2+m_3)^2 + \frac{1+q}{3}\left(\sum_{i=0}^{3} m_i^2\right)$$

- Asymptotically free theory with $N_f = 3$

precise relationship between $u$ and $Tr\phi^2$ Seiberg-Witten

$$u = \langle Tr\phi^2 \rangle - \frac{1}{6}(\vartheta_4^4 + \vartheta_3^4)\sum_{i=0}^{3} m_i^2$$

Recall

$$m_\pm = m_2 \pm m_0, \quad \tilde{m}_\pm = m_3 \pm m_1,$$

We take the limit

$$\tilde{m}_- \to \infty, \quad q \to 0,$$

with

$$\tilde{m}_- q = \Lambda^3 \qquad \text{fixed}$$

Matrix action reduces to

$$W(M) = \tilde{m}_+ \log M - \frac{\Lambda_3}{2M} + m_2 \log(M-1).$$

Spectral curve for $N_f = 3$ theory becomes

$$x^2 = \frac{\Lambda_3^2}{4z^4} - \frac{\tilde{m}_+\Lambda_3}{z^3(z-1)} - \frac{u - (m_2 + \frac{1}{2}\tilde{m}_+)\Lambda_3}{z^2(z-1)} + \frac{m_0^2}{z(z-1)} + \frac{m_2^2}{z(z-1)^2} - \frac{m_2\Lambda_3}{z^2(z-1)}.$$

Free energy and discriminant of the model agrees completely with that of the standard SW curve

$$
\begin{aligned}
y^2 \;=\; & x^2(x-u) - \frac{1}{4}\Lambda_3^2(x-u)^2 \\
& -\frac{1}{4}(m_+^2 + m_-^2 + \tilde{m}_+^2)\Lambda_3^2(x-u) + m_+ m_- \tilde{m}_+ \Lambda_3 x \\
& -\frac{1}{4}(m_+^2 m_-^2 + m_-^2 \tilde{m}_+^2 + \tilde{m}_+^2 m_+^2)\Lambda_3^2
\end{aligned}
$$

- Asymptotically free theory with $N_f = 2$

Spectral curve:

$$
x^2 = \frac{\Lambda_2^2}{4z^4} + \frac{\tilde{m}_+\Lambda_2}{z^3} + \frac{u}{z^2} + \frac{m_+\Lambda_2}{z} + \frac{\Lambda_2^2}{4}
$$

Matrix action:

$$W(M) = \tilde{m}_+ \log M - \frac{\Lambda_2}{2M} - \frac{\Lambda_2 M}{2}$$

Predicts the same free energy and discriminant as the SW curve

$$y^2 = (x^2 - \frac{1}{4}\Lambda_2^4)(x-u) + m_+\tilde{m}_+\Lambda_2^2 x - \frac{1}{4}(m_+^2 + \tilde{m}_+^2)\Lambda_2^4$$

## ♠ Mysteries

- Integration contour, range of integration
- What about $N_f = 0, 1$ ?

- Another matrix model: generalization of $\mathrm{CP}^1$ model with action

$$TrM(\log M - 1) \qquad \text{Klemm,Sulkowski}$$