Thermal gauge theory + blackhole-string transition in $AdS_5 \times S^5$

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Yoneya Fest

String theory, GR, Blackholes

- Yoneya (and Scherk + Schwarz) discovered that string theory contains Einstein's theory of gravity.

- Einstein's theory has black hole solutions.

- In the past decade string theory (D-branes) has led to an understanding of various issues in black hole physics.

$$\overset{\text{Gravity}}{S_{\text{Hawking-Bek.}}} = \overset{\text{gauge theory}}{S_{\text{Boltzmann}}} \equiv \ln \Omega$$

Hawking radiation

(e.g. D1-D5 system)

$\Downarrow$

Superconformal theory in 2-dim.

valid for both weak + strong coupling

Non-supersymmetric bhs are more difficult

Small Schwarzschild (Sch.) bh in 10-dim

We want to study the blackhole $\to$ string transition (Horowitz-Polchinski) for the Small Sch. bh. in 10-dim.

10-dim. bh :

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_8^2$$

$$f(r) = 1 - \frac{r_+^7}{r^7}\,, \qquad r_+^7 = 2 G_{10} M$$

$$T \sim \frac{1}{r_+} = \frac{1}{(2G_{10}M)^{1/7}}$$ —ve specific heat

As we scale the temp. up, r_+ eventually approaches $l_s \equiv \sqrt{\alpha'}$, the string length.

Supergravity not valid + the 'blackhole' should be described in terms of degrees of freedom relevant to the string scale

Horowitz-Polchinski $\Rightarrow$

transition (Crossover) is smooth.

Heuristic discussion:

- $S_{BH} \sim \frac{r_+^8}{G_{10}} \sim M^{8/7} G_{10}^{1/7} \equiv (M \ell_p)^{8/7}$

  $S_{\text{Hagedorn}} \sim M \ell_s$ $\qquad \ell_p^8 = g_s^2 \ell_s^8$

  These can be matched when: $\boxed{M \ell_s g_s^2 \sim 1}$

- In terms of the temperature: $\boxed{T \ell_s \sim 1}$

  If you tune the temperature to a critical temperature $T_c \sim \frac{1}{\ell_s}$, then there is a smooth transition (entropy, mass ..vary smoothly)

- All estimates above are crude as when $T_c \sim \frac{1}{\ell_s}$ neither supergravity nor the free string spectrum are valid. (large curvature / $g_s^2 \sim \frac{1}{M \ell_s} \neq 0$)

Summary:

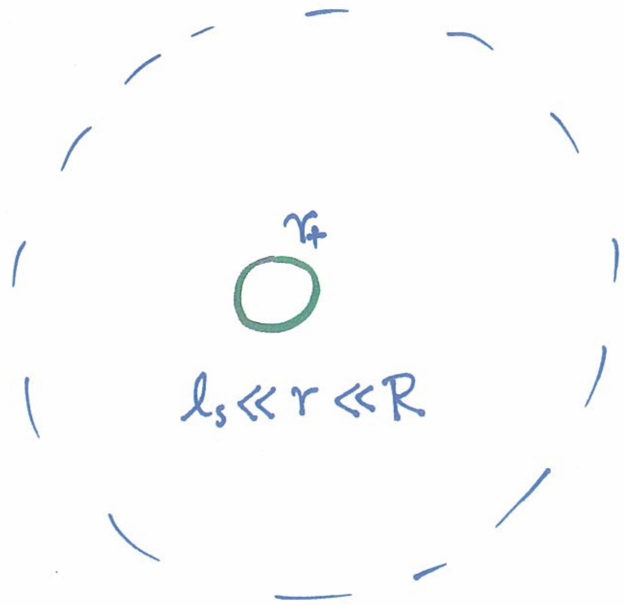
- Hor.-Pol. $\Rightarrow$ as $T \to \frac{t_c}{l_s}$, $(t_c \sim 1)$

  bh. entropy and other thermodynamic quantities match with the entropy + thermodynamics of a highly degenerate state of strings of a similar mass. This transition is smooth + it is more appropriate to call it a 'crossover.'

- We would like to discuss this process in the $\mathcal{N}=4$, $SU(N)$ SYM on $S^3 \times S^1$ using the AdS/CFT correspondence at finite temperature.

- For this we will need to discuss the embedding of the 10-dim. Sch. bh. in $AdS_5 \times S^5$ (the asymptotic metric is $AdS_5 \times S^5$ rather than R^{10}).

Horowitz + Hubeny:



$r >> R$
standard
$AdS_5 \times S^5 +$
$F^{(5)}$

The key point is that the solution is such that the Sch. bh. remains small (horizon area does not change). The 5-form does not fall through and cause the bh. to grow.

Summary:

Small 10-dim. Sch. bh. can be embeded in a spacetime with $AdS_5 \times S^5$ boundary conditions.

(Note: the bh metric is uniform over S^3 on the boundary of AdS_5.)

We can now hope to use the thermal gauge theory on S^3 to study the bh $\to$ string transition provided we can identify the corresponding transition in the gauge theory.

Recall that there are several known transitions in the bulk IIB string theory:

1. Blackhole nucleation. $T \sim \frac{1}{R}$.
   2nd order.

2. Hawking-Page. $T \sim \frac{1}{R}$. 1st order

3. Horowitz-Polchinski. $T \sim \frac{1}{\ell_s}$. higher order

4. Hagedorn. $T \sim \frac{1}{\ell_s}$

What is the correspondence in the thermal gauge theory.

Thermal Gauge theory effective action:

$\mathcal{N}=4$ SU(N) gauge theory on $\underset{R}{S^3} \times \underset{T^{-1}}{S^1}$

Field content A_i, A_0, ϕ^a, ψ^a

$\phi^a \to$ massive on S^3.

Order parameter: $U(\vec{x}) = \exp\left(i\int_0^\beta A_0(\vec{x},\tau)d\tau\right)$

e.g. at $g_{YM}=0$,

$$Z \simeq \int dU \, e^{\omega_1(\beta)\,|\mathrm{tr}\,U|^2}$$

$U = e^{iA}$, $A \equiv$ zero mode of $A_0(\vec{x})$.

In general (for more than one reason)

$$Z \sim \int dU \, e^{S_{eff}(U)}$$

$S_{eff}(U)$ is a multinomial in $\frac{1}{N}\mathrm{tr}\,U^{n_k}$ with terms of the form

$$\frac{\mathrm{tr}\,U^{n_1}}{N}\frac{\mathrm{tr}\,U^{n_2}}{N}\cdots\frac{\mathrm{tr}\,U^{n_p}}{N} \, , \quad \sum_p n_p = 0 \bmod (N)$$

$Z(N)$ sym.

Phase transitions at large N:

To set the ideas and illustrate the various phase transitions at large N, let us discuss a toy model:

$$S_{eff}(U) = a\,|\mathrm{tr}\,U|^2 + \frac{b}{N^2}|\mathrm{tr}\,U|^4$$

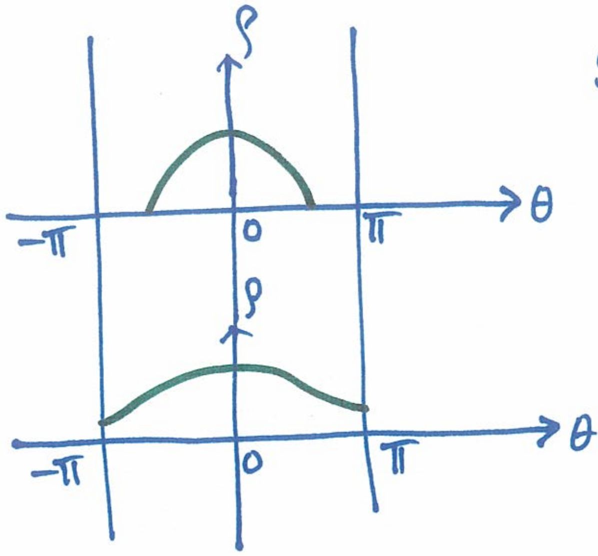
a and b are functions of the temperature T.

choose $b>0$.

Mean field theory analysis:

$$\rho(\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i), \quad \rho_1 \equiv \frac{1}{N}\mathrm{tr}\,U = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\,e^{i\theta}\rho(\theta)$$

$N\to\infty$, $\rho(\theta)$ is continous and it has 2 distinct behaviors as a function on a circle.



gapped

3rd order transition*

ungapped.

* GWW

Effective Action (mean field):

$$S(\rho_1) = \begin{cases} -(a\rho_1^2 + b\rho_1^4) + \frac{1}{2}\ln 2\rho_1(1-\rho_1) + \frac{1}{4} & \frac{1}{2} \le \rho_1 \le 1 \quad \text{(Gapped)} \\ (1-a)\rho_1^2 - b\rho_1^4 & 0 \le \rho_1 \le \frac{1}{2} \quad \text{(ungapped)} \end{cases}$$

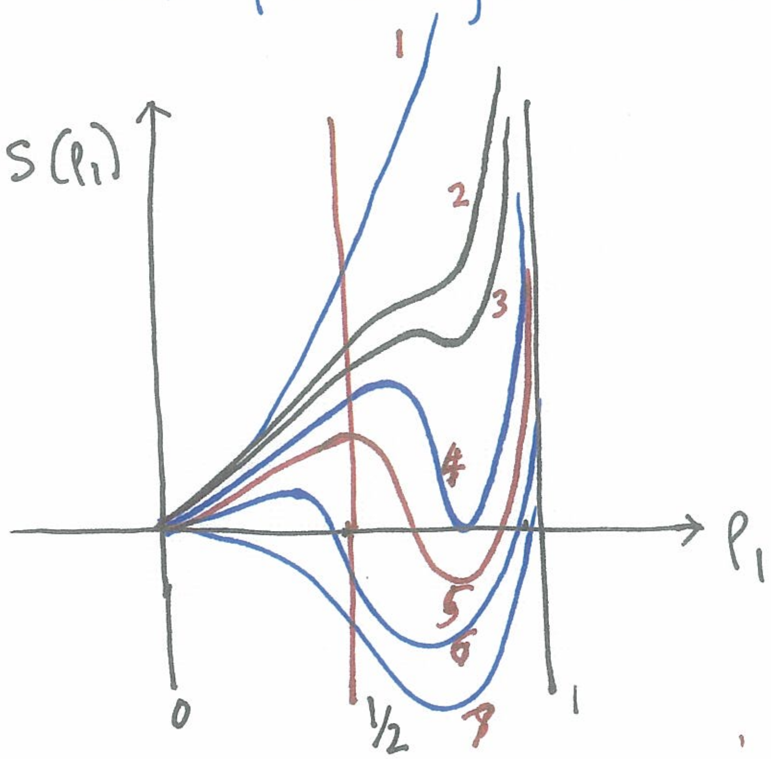
$S(\rho_1)$, $S'(\rho_1)$, $S''(\rho_1)$ are continous

at $\rho_1 = \frac{1}{2}$

$S'''(\rho_1)$ is discontinous

at $\rho_1 = \frac{1}{2}$

$S(\rho_1)$ plots for various (a,b)



$1 \to \cancel{8}\,7$ is $\underline{a}$ thermal history

1. min. at $\rho_1 = 0$	AdS_5
2. pt. of inflexion	bh. nucleation
3. new local min. + max	$\downarrow$
4. first order	Hawking-Page
~~5. change of dominance~~	$\downarrow$
$\boxed{\text{5. 3rd order transition}}$	$\boxed{\text{Horowitz-Polchinski}}$
$a + \frac{b}{2} = 1$	$\downarrow$
6. 3 extrema	
7. $\rho_1 = 0$ unstable	Hagedorn.

- A detailed quantitative theory of the thermal history would require a computation of the coeff. of the effective lagrangian at $\lambda = g^2_{YM} N \gtrsim 1$. This is very difficult.

- However there is one transition (the 3rd order large N transition) for which the $O(1)$ quantum partition function can be exactly computed in a scaling limit.

- For the toy model $\rho_1 = \frac{1}{2}$ is a saddle point iff $a + \frac{b}{2} = 1$.

- We next turn to some technical results that enable this.

A useful formula:

$$S_{eff}(U) = N^2 \sum_{i=1}^{P} a_i \left| \frac{tr}{N} U^i \right|^2 + N^2 P\left(\frac{tr}{N} U^i, \frac{tr}{N} U^{-i}\right)$$

P is a multinomial

$$Z = \int dU \, e^{S_{eff}(U)}$$

$$= \left(\frac{N^4}{2\pi^2}\right)^P \int \prod_{i=1}^{P} dg_i \, d\bar{g}_i \, d\mu_i \, d\bar{\mu}_i \; e^{\tilde{S}}$$

$$\tilde{S} = S_{eff}(i\mu_i, -i\bar{\mu}_i) + i N^2 \sum_i (g_i \bar{\mu}_i + \bar{g}_i \mu_i)$$

$$+ N^2 F(\bar{g}_i, g_i)$$

where $N^2 F(\bar{g}_i, g_i)$ is the free energy of the single trace partition function.

$$\mathcal{Z} = \int dU \; e^{N \sum_{i \geq 1} (g_i \, tr U^i + c.c.)}$$

Toy Model :

e.g. $S_{eff} = a|\mathrm{tr}\,U|^2 + \frac{b}{N^2}|\mathrm{tr}\,U|^4$

$$\tilde{S} = N^2\left(-a|\mu|^2 + b|\mu|^4\right)$$

$$+ iN^2(g\bar{\mu} + \bar{g}\mu) \quad + N^2 F(g,\bar{g})$$

The phase of g can be absorbed in the measure : $g = |g|e^{i\alpha}$

$$\tilde{S} = N^2\left(-a|\mu|^2 + b|\mu|^4\right) + i|g|(\mu+\bar{\mu})N^2$$

$$+ N^2 F(|g|).$$

Call $|g| \equiv g$

$$N^2 F(g) = \begin{cases} \dfrac{N^2 g^2}{4} + e^{-Nf(g)}\left(\dfrac{F_1^{(1)}(g)}{N} + \dfrac{F_2^{(1)}(g)}{N^2} + \cdots\right) & g \leq 1. \\[2ex] \dfrac{N^2 g^2}{4} + F_0^{(2)}(g) + \dfrac{1}{N^{2/3}} F_1^{(2)}(g) + \cdots & g = 1 + tN^{-2/3} \\[2ex] N^2\left(g - \frac{1}{2}\ln g - \frac{3}{4}\right) + F_0^{(3)}(g) + \frac{1}{N^2} F_1^{(3)}(g) & g \gtrsim 1 \end{cases}$$

a) $f(g) = -\ln g + \ln\left(1+\sqrt{1-g^2}\right) - \sqrt{1-g^2}$

b) $F_n^{(1)} \sim \left[\dfrac{1}{(1-g)^{3/2}}\right]^n$

$$F_n^{(3)} \sim \left[\frac{1}{(g-1)^{3/2}}\right]^{2n}$$

divergent as $g \to 1$

c) $F_0^{(2)}\left(g = 1 + t N^{-2/3}\right) \equiv F(t)$

$$\ddot{F}(t) = -f^2(t)$$

$$\frac{1}{2}\ddot{f} = tf + f^3 \quad \text{(Painleve II).}$$

d)

$$F(t) = \begin{cases} \dfrac{t^3}{6} - \dfrac{1}{8}\ln(-t) - \dfrac{3}{128}\dfrac{1}{t^3} \cdots & -t \gg 1 \\[2ex] \dfrac{1}{2\pi} e^{-\frac{4\sqrt{2}}{3}t^{2/3}} \left(-\dfrac{1}{8\sqrt{2}}\dfrac{1}{t^{3/2}} + \dfrac{35}{384}\dfrac{1}{t^3} + \cdots\right), & t \gg 1 \end{cases}$$

Evaluating the integral:

Now check if the critical point $|g| = 1$ is a saddle point of the $O(N^2)$ action:

$$\tilde{S} = N^2\left(-a|\mu|^2 + b|\mu|^4\right) + i|g|(\mu+\bar{\mu})N^2 + \frac{N^2}{4}|g|^2$$

Yes it is iff $a + \frac{b}{2} = 1$.

Then we perform gaussian integral around the μ, g saddle points: $\left(|g|=1,\ |\mu| = -\frac{i}{2},\ \theta = 0\right)$

$$g = 1 + t N^{-2/3}, \qquad |\mu| = -\frac{i}{2} + n N^{-4/3}$$

$$a = a(T^c) + a'(T^c)(T - T_c)$$

$$b = b(T^c) + b'(T^c)(T - T_c)$$

$$\frac{T - T_c}{T_c} = q N^{-2/3}, \qquad q \equiv \text{fixed.}$$

$$Z \simeq iN\sqrt{\frac{\pi}{b}}\; e^{-N^2\frac{(1-a(T_c))^2}{4b(T_c)} + F(c_1 q)}$$

$F(c_1 q)$ is given by the Painleve function.

$$c_1 = \left(\frac{2a'+b'}{b}\right)_{T=T_c}$$

It is known that $F(t \equiv c_1 q)$ is a smooth function of 't'.

(hep-th/050227 : Alvarez-Gaumé, Cesar Gomez, Hong Liu + SRW).

The general problem was solved in

(hep-th/0605041 : Alvarez-Gaumé, Pallab Basu, Marcos Merino, SRW).

The general case :

- Since the thermal history is parametrized by the temp. T, we will discuss the first critical point

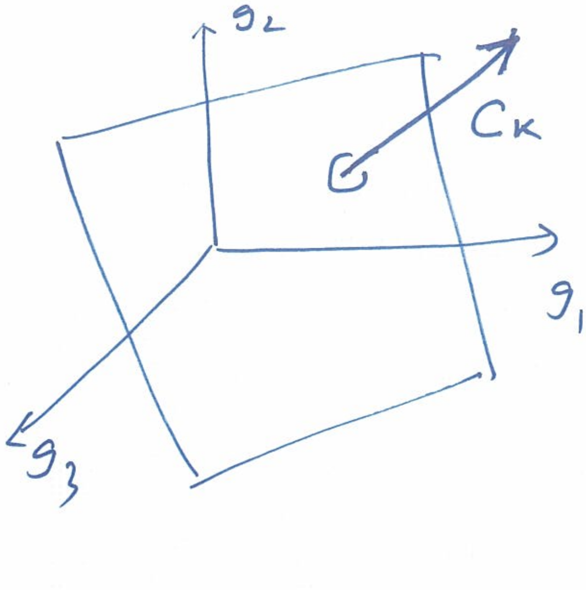
$$\rho(\pi-\theta) \sim (\theta-\pi)^{2k}, \quad k=1$$

- The critical surface where the gap in the eigenvalue distribution opens is given by a plane in the space

$\mathcal{g} = \{g_k\}$

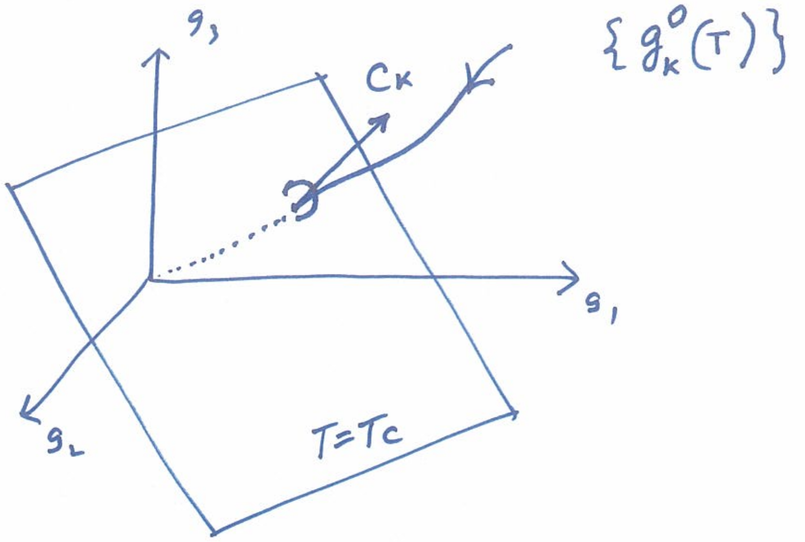
$$\sum_{k=-\infty}^{+\infty} C_k \, k^2 \left(g_k^c + \bar{g}_k^c \right) = -1$$

$$C_k = \frac{(-1)^k}{k}.$$



• The key question is whether the $O(N^2)$ action has a saddle point which lies on the critical surface.

We will <u>assume</u> that this is true



$$g_K = g_K^c + t_K N^{-2/3}$$

$$\mu_K = \mu_K^c + n_K N^{-4/3}$$

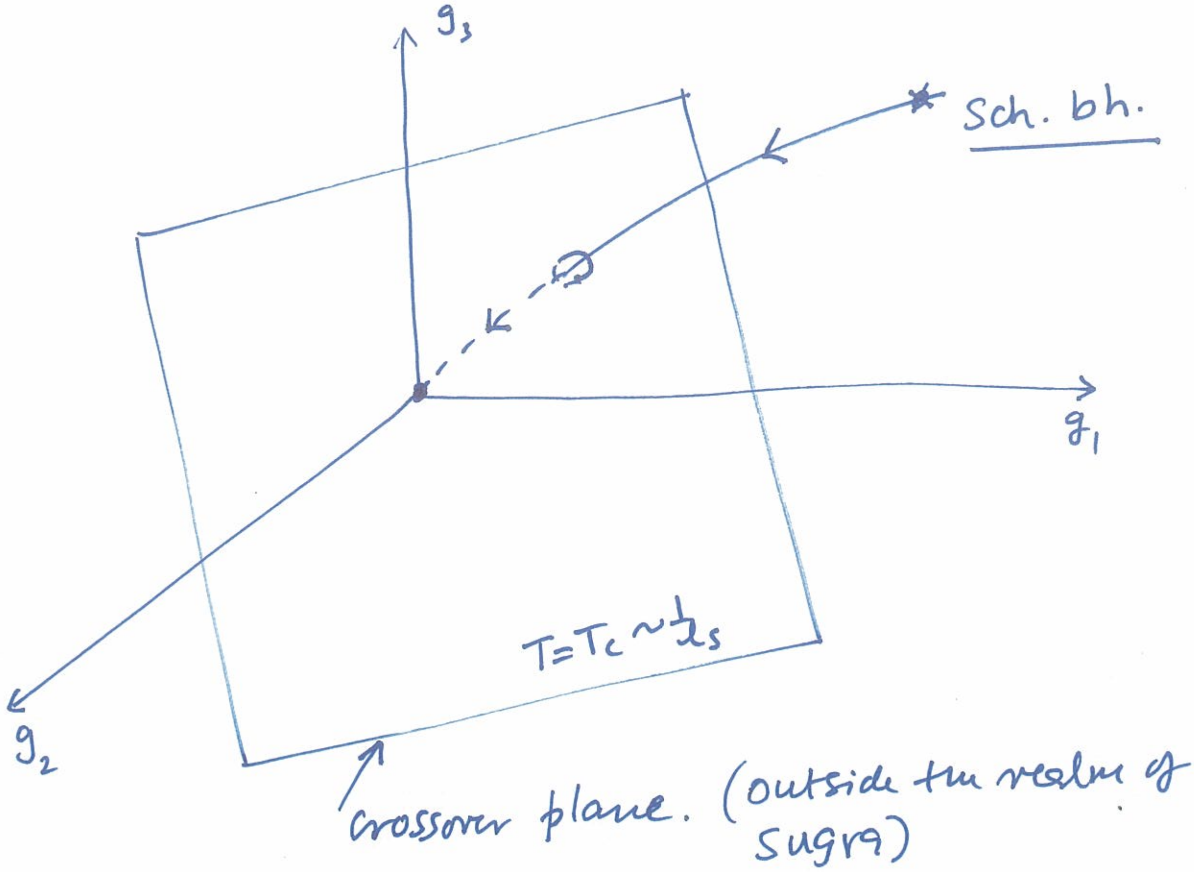
$$A = A^c + \left(\frac{\partial A}{\partial T}\right)_c (T - T_c)$$

$$\left(\frac{T-T_c}{T_c}\right) = \tilde{g} N^{-2/3}$$

$O(1)$ free energy: $F(\tilde{g}\, \underbrace{\vec{C}}_{} \cdot \tilde{\vec{t}}) \equiv F(t)$

parameters at $T=T_c$.

Applications to the bh - string cross over :



- Sch. bh (a sugra solution) corresponds to the gapped phase of the gauge theory. This can be seen by studying the formulas for $\tilde{S}$ in the various phases. Pg. 13.

- as $T_{BH} \to \frac{1}{l_s}$ the thermal history approaches the critical plane, where there is a cross over.

In summary:

- We identified the Horowitz-Polchinski cross over with the cross over at the critical plane of the large N phase transition.

- The cross over is smooth in the double scaled variable $\left(\frac{T-T_c}{T_c}\right) N^{2/3}$.

- The entire discussion is in the euclidean metric both in the gauge theory and the bulk.

  Implication for the Lorentzian bh?

- F(t) is also the free energy of the OB theory in 0+1 dim. (Klebanov-Maldacena-Seiberg).

  Gives a hint to an exact bulk description.