Fiber bundles and matrix models
- gauge/gravity correspondence for
  SU(2|4) theories -

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Introduction
• **Emergence of space-time** is one of the key concepts of matrix models as nonperturbative definition of superstring e.g.) Matrix theory BFSS IIB matrix model IKKT

• Eigenvalues of matrix in c=1 matrix model ~ Liouville direction Das-Jevicki, Gross-Klebanov, Polchinski

• Bubbling AdS geometry LLM, Jevicki’s talk

• It should also be useful to study relationship between field theories and matrix models

• large N reduction Eguchi-Kawai

• quenched reduced model Bhanot-Heller-Neuberger, Parisi, Gross-Kitazawa twisted reduced model Gonzalez-Arroyo-Okawa ~ noncommutative field theory

• **T-duality for D-brane effective theories** `matrix T-duality’ Taylor

• It is important to understand how curved space-time is realized in matrix models ~ our problem cf.) Hanada-Kawai-Kimura
Here we focus on the gauge/gravity correspondence for the SU(2|4) theories proposed by Lin-Maldacena

1) N=4 SYM on $R \times S^3/\mathbb{Z}_k$
   \[ \downarrow \text{dim. reduction} \]

2) 2+1 SYM on $R \times S^2$
   \[ \downarrow \text{dim. reduction} \]

3) plane wave matrix model
   \[ \text{BMN} \]

\text{doubly interesting}

- different vacua of one theory $\sim$ different geometries
- suggest relations between fields theories on curved spaces and matrix model \(\text{`inverse'}\) of dimensional reductions

We show these relations directly on gauge theory side
SU(2|4) theories and gauge/gravity correspondence
SU(2|4) theories

Start with N=4 SYM on \( \mathbb{R} \times S^3 \)

\[
SU(2,2|4) \supset SO(2,4) \times SO(6) \supset \mathbb{R} \times SO(4) \times SO(6)
\]

\[
SO(4) = SU(2)_L \times SU(2)_R
\]

Divide by subgroup of SU(2)_R \( \xrightarrow{\text{SU(2|4) theories}} \)

16 supercharges

1) N=4 SYM on \( \mathbb{R} \times S^3 / \mathbb{Z}_k \)

\[
S_{S^3/\mathbb{Z}_k} = \int dt d\Omega_3 \text{Tr} \left( \frac{1}{4} F_{ab}^2 + \cdots \right)
\]

2) 2+1 SYM on \( \mathbb{R} \times S^2 \)

\[
S_{S^2} = \int dt d\Omega_2 \text{Tr} \left( -\frac{1}{2} (f_{12} - \phi)^2 - \frac{1}{2} (D_a \phi)^2 + \cdots \right)
\]

Dim. reduction

3) plane wave matrix model

Kim-Klose-Plefka

\[
S_{PW} = \text{Tr} \left( -\frac{1}{2} (Y_i - \frac{i}{2} \epsilon_{ijk} [Y_j, Y_k])^2 + \cdots \right)
\]

Common features: mass gap, discrete spectrum, many discrete vacua
Gravity duals

Lin-Maldacena

general form of smooth solutions of type IIA SUGRA that have SU(2|4) isometry

\[
d_{10}^2 = \left( \frac{\ddot{V} - 2\dot{V}}{-V''} \right) \left\{ -4 \frac{\dddot{V}}{V - 2\dot{V}} dt^2 + \frac{-2V''}{V} (d\rho^2 + d\eta^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta} d\Omega_2^2 \right\}
\]

\[\Delta = (\ddot{V} - 2\dot{V})V'' - (\dot{V}')^2, \quad . = \rho \partial_{\rho}, \quad ' = \partial_{\eta}\]

\(V(\rho, \eta)\): electrostatic potential for axially symmetric system

# of conducting disks \(T \sim\) topology

\(Q_s\): D2-brane charge

\(d_s\): NS 5-brane charge

asymptotic behavior of \(V\) specifies a theory

config. of disks specifies a vacuum
1) SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$

holonomy around generator of $\pi_1(S^3/\mathbb{Z}_k) = \mathbb{Z}_k$

$$W = \text{diag}(e^{i\frac{2\pi}{k}\beta_1}, \ldots, e^{i\frac{2\pi}{k}\beta_1}, e^{i\frac{2\pi}{k}\beta_2}, \ldots, e^{i\frac{2\pi}{k}\beta_2}, \ldots, e^{i\frac{2\pi}{k}\beta_T}, \ldots, e^{i\frac{2\pi}{k}\beta_T})$$

2) SYM on $\mathbb{R} \times S^2$  

Dirac monopoles

$$\phi = \text{diag}(q_1, q_1, \ldots, q_1, q_2, q_2, \ldots, q_2, \ldots, q_T, q_T, \ldots, q_T)$$

$$A_\theta = 0$$

$$A_\varphi = \begin{cases} 
(1 - \cos \theta) \phi & \text{for } 0 \leq \theta < \pi \\
(-1 - \cos \theta) \phi & \text{for } 0 < \theta \leq \pi 
\end{cases}$$
3) PWMM

concentric fuzzy spheres with different radii

\[ Y_i = - \begin{pmatrix} L_i^{[j_1]} & N_1 \\ \vdots & \vdots \\ L_i^{[j_{\ell}]} & \vdots & N_T \end{pmatrix} \]

\[ L_i^{[j_s]} : \text{spin } j_s \text{ representation of } SU(2) \]

\[ \eta \]

\[ \begin{align*}
2j_T + 1 & \quad N_T \\
2j_2 + 1 & \quad N_2 \\
2j_1 + 1 & \quad N_1 \\
0 &
\end{align*} \]

infinitely large disk
PWMM vs SYM on $\mathbb{R} \times S^2$
theory around

\[ Y_i = \begin{pmatrix} L_{i1}^{[q_1]} \cdots N_1 \\ \vdots \\ L_{i[1]}^{[q_T]} \cdots N_T \\ \vdots \\ L_{i[T]}^{[q_T]} \cdots N_T \end{pmatrix} \]

\[ \phi = \text{diag}(q_1, \ldots, q_1, \ldots, q_T, \ldots, q_T) \]

\[ A_\theta = 0 \]

\[ A_\varphi = \begin{cases} (1 - \cos \theta) \phi & \text{for } 0 \leq \theta < \pi \\ (-1 - \cos \theta) \phi & \text{for } 0 < \theta \leq \pi \end{cases} \]
Outline of proof

See agreement of harmonic expansions of the theories around the vacua of two theories including int. terms

monopole harmonics by Wu-Yang

\[ Y_{jm}^{q} (\theta, \varphi) \quad (|q| \leq J) \]

\[ j, j' \to \infty, j - j' = q \]

`fuzzy' spherical harmonics

\[ Y_{jm}^{jj'} \sim \sum_{rr'} C_{j_{r} j_{r'}}^{j_{m}} |j_{r} \rangle \langle j_{r}' - r'| \quad (|j - j'| \leq J \leq j + j') \]

~basis for rectangular matrices

Topologically nontrivial configuration

continuum limit of concentric fuzzy spheres with different radii

= monopoles on \( S^2 \)

SYM on $\mathbb{R} \times S^2$ vs SYM on $\mathbb{R} \times S^3/Z_k$
SYM on $\mathbb{R} \times S^2$ \[ \eta \]

\[ \begin{array}{c}
N \\
\downarrow k \\
N \\
\downarrow k \\
N \\
\downarrow \longrightarrow \rho
\end{array} \]

SYM on $\mathbb{R} \times S^3/Z_k$ \[ \eta \]

\[ \begin{array}{c}
N \\
\downarrow k \\
\longrightarrow \rho
\end{array} \]

\textbf{theory around} $-\infty < s < \infty$

$\phi = k \text{diag} \left( \ldots, \frac{s-1}{2}, \ldots, \frac{s-1}{2}, \frac{s}{2}, \ldots, \frac{s}{2}, \ldots \right)$

$A_\theta = 0$

$A_\varphi = \begin{cases} 
(1 - \cos \theta) \phi & 0 \leq \theta < \pi \\
(-1 - \cos \theta) \phi & 0 < \theta \leq \pi 
\end{cases}$

with periodicity $X^{s+1, t+1} = X^{s, t}$

\[ \square \]

theory around trivial vacuum

\[ \square \]

extension of matrix T-duality to nontrivial fiber bundle $S^3/Z_k$ as $S^1/Z_k$ on $S^2$
Extension of matrix T-duality
Matrix T-duality on principal $S^1$ bundle

$S^3$ as $S^1$ on $S^2$ (Hopf bundle)

metric of $S^3$
\[ ds^2_{S^3} = d\theta^2 + \sin^2 \theta d\varphi^2 + (d(\psi \pm \varphi) + (\cos \theta \pm 1) d\varphi)^2 \]

fiber direction
\[ y_{[1]} = \psi + \varphi \quad \text{for } 0 \leq \theta < \pi \quad \text{patch 1} \]
\[ y_{[2]} = \psi - \varphi \quad \text{for } 0 < \theta \leq \pi \quad \text{patch 2} \]

connection 1-form
\[ \omega = d(\psi \pm \varphi) + (\cos \theta \mp 1) d\varphi \]

vertical-horizontal decomposition of tangent space of $S^3$

fiber direction
\[ \omega(E_\alpha) = 0 \quad (\alpha = 1, 2) \]

\[ S_{S^3} = \int d\Omega_3 \text{Tr} \left( \frac{1}{4} F_{ab}^2 \right) \quad \text{---->} \quad S_{S^2} = \int d\Omega_2 \text{Tr} \left( \frac{1}{2} (f_{12} - \phi)^2 + \frac{1}{2} (D_\alpha \phi)^2 \right) \]

\[ A_\alpha = a_\alpha \quad (\alpha = 1, 2) \]
\[ A_3 = \phi \]
matrix T-duality between $S^2$ and $S^3$

theory around the vacuum of $S_{S^2}$ with $U(N \times \infty)$ and periodicity
$= S_{S^3}$ with $U(N)$

$$U(N) \rightarrow U(N \times \infty)$$

vacuum

$\hat{\phi} = \frac{1}{2}(\cdots, s - 1, s, s + 1, \cdots)$

$\hat{a}_1 = 0$

$\hat{a}_{2}^{[1]} = \tan \frac{\theta}{2} \hat{\phi}, \quad \hat{a}_{2}^{[2]} = - \cot \frac{\theta}{2} \hat{\phi}$

periodicity for fluctuations

$\tilde{a}_{[I]}^{[l]}(s,t) = \tilde{a}_{[I]}^{[l]}(s-t), \quad \tilde{\phi}_{[I]}(s,t) = \tilde{\phi}_{[I]}(s-t)$

Fourier transf. on each patch (T-duality)

$$A_{\alpha} = \sum_{s} \tilde{a}_{\alpha}^{[1,2]}(s) e^{-\frac{i}{2}s(\psi \pm \varphi)}$$

$A_3 = \sum_{s} \tilde{\phi}^{[1,2]}(s) e^{-\frac{i}{2}s(\psi \pm \varphi)}$

monopole charge = momentum

relation between monopole harmonics and harmonics on $S^3$

/\text{Z}_k\text{ easy}
We can extend matrix T-duality to that on general $S^1$ principal bundles.

Ex.)

$S^5$ as $S^1$ on $\mathbb{CP}^2$ (gravitational and electromagnetic instanton)
Heisenberg nilmanifold as $S^1$ on $T^2$ (constant magnetic flux on $T^2$)
Conclusion & Discussions
• We showed

theory around each vacuum of SYM on R x S^3/Z_k
= theory around a vacuum of SYM on R x S^2 with periodicity

theory around each vacuum of SYM on R x S^2
= theory around a vacuum of PWMM

• Nontrivial check of gauge/gravity correspondence for SU(2|4) theories

• Continuum limit of concentric fuzzy spheres~ monopoles on S^2

• Extension of matrix T-duality to that on S^1 principal bundles

• Theory around every vacuum of SU(2|4) theories is realized in PWMM

• N=4 SYM on R x S^3 is realized in PWMM

• S^3 is realized in terms of three matrices \ Y_1 \ Y_2 \ Y_3 \n
\[ Y_i = - \begin{pmatrix} \cdot \cdot \cdot \ L_i^{[j_s-1]} \ 
L_i^{[j_s]} \ 
L_i^{[j_s+1]} \ 
\cdot \cdot \cdot \end{pmatrix} \otimes 1_N \]
Outlook

• CS on $S^3$ ↔ pure Yang Mills on $S^2$ with K. Ohta
• Nonabelian case e.g.) $S^7$ as $S^3$ on $S^4$, $SU(3)$ as $U(2)$ on $CP^2$
• Lattice theory for N=4 SYM on $R \times S^3$ cf.) Kaplan
• Relate our findings to (flux) compactification in string theory
Dimensional reduction

U(N) YM on $\mathbb{R}^p \times S^1$

$S_{p+1} = \int d^{p+1}x \frac{1}{4} \text{Tr}(F_{MN}^2)$

$x^M = (x^\mu, y) \quad \mu = 1, \ldots, p$

$y \sim S^1$ with radius $R$

D$p$-brane

\[ A_\mu = a_\mu \]
\[ A_y = \phi \]

U(N) YM with Higgs on $\mathbb{R}^p$

$S_p = \int d^p x \text{Tr} \left( \frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} (D_\mu \phi)^2 \right)$

D(p-1)-brane

\[ U(N) \rightarrow U(N \times \infty) \]

\[ \cdots \]

\[ (s, t) \]

\[ N \]

\[ N \]

\[ \cdots \]
S\(^1\) compactification

S\(^1\) with radius \(\tilde{R}\)

\[\phi = \tilde{\phi} + \tilde{\phi} \quad \tilde{\phi} = 2\pi \tilde{R} \text{diag}(\cdots, s-1, s, s+1, \cdots) \otimes 1_N\]

\[a_\mu^{(2)}(x, y) = a_\mu^{(0)}(x, y) e^{-i\frac{s}{\tilde{R}} y}\]

\[A_\mu(x, y) = \sum_s a_\mu^{(s)}(x) e^{-i\frac{s}{\tilde{R}} y}\]

\[A_y(x, y) = \sum_s \tilde{\phi}^{(s)}(x) e^{-i\frac{s}{\tilde{R}} y}\]

T-duality

winding #: \[s - t\]

momentum: \[s - t\]

recovery of Sp+1