Komaba 2007 on the occasion of Prof. Yoneya's 60 th Birthday

# Entanglement Entropy
# and
# AdS/CFT Correspondence

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Based on hep-th/0603001 [PRL 96(2006)181602]
hep-th/0605073 [JHEP 0608(2006)045]
hep-th/0608213 [To appear in JHEP]
hep-th/0611035 [JHEP 0701(2007)090]
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# ① Introduction

In quantum mechanical systems, it is very useful to measure how entangled ground states are.

An important measure is the entanglement entropy.

[Various Applications]

- Quantum Information and quantum computing
- Condensed matter physics (EE = a new order parameter?)
- Black hole physics (EE = the origin of BH entropy?)

→ Why not, in String theory?

## Definition of entanglement entropy

Divide a given quantum system into two parts A and B.

Then the total Hilbert space becomes factorized

$$H_{tot} = H_A \otimes H_B \quad .$$

We define the reduced density matrix $\rho_A$ for A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

taking trace over the Hilbert space of B .

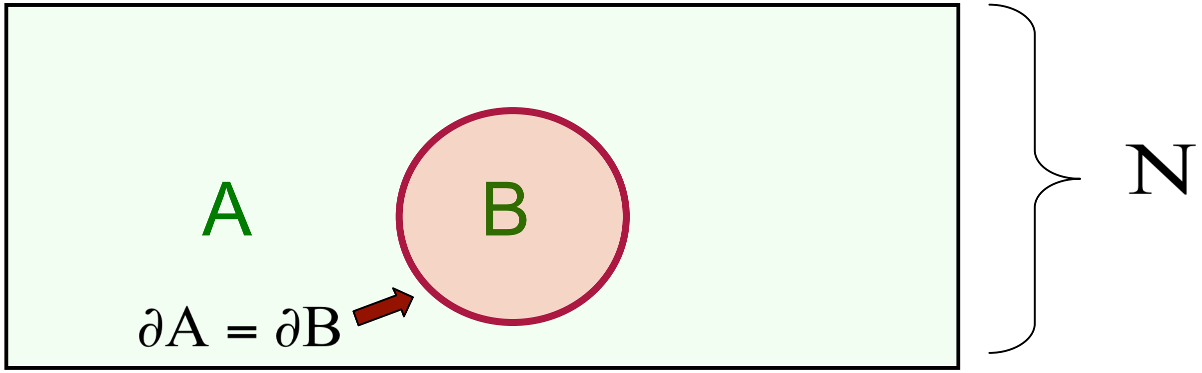
Now the entanglement entropy $S_A$ is defined by the von Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \qquad .$$

In this talk we consider the entanglement entropy in quantum field theories on (d+1) dim. spacetime

$$M = R_t \times N .$$

Then, we divide $N$ into A and B by specifying the boundary $\partial A = \partial B \subset N$ .



E.E. is the entropy for an observer who is only accessible to the subsystem A and not to B.

This setup is very similar to black holes !

→ This property suggests its gravity dual description.

Motivated by this, we would like to apply AdS/CFT to find a holographic dual of entanglement entropy.

## Other Useful Properties

(i) E.E. is a sort of a `non-local version of correlation functions'. (cf. Wilson loops)

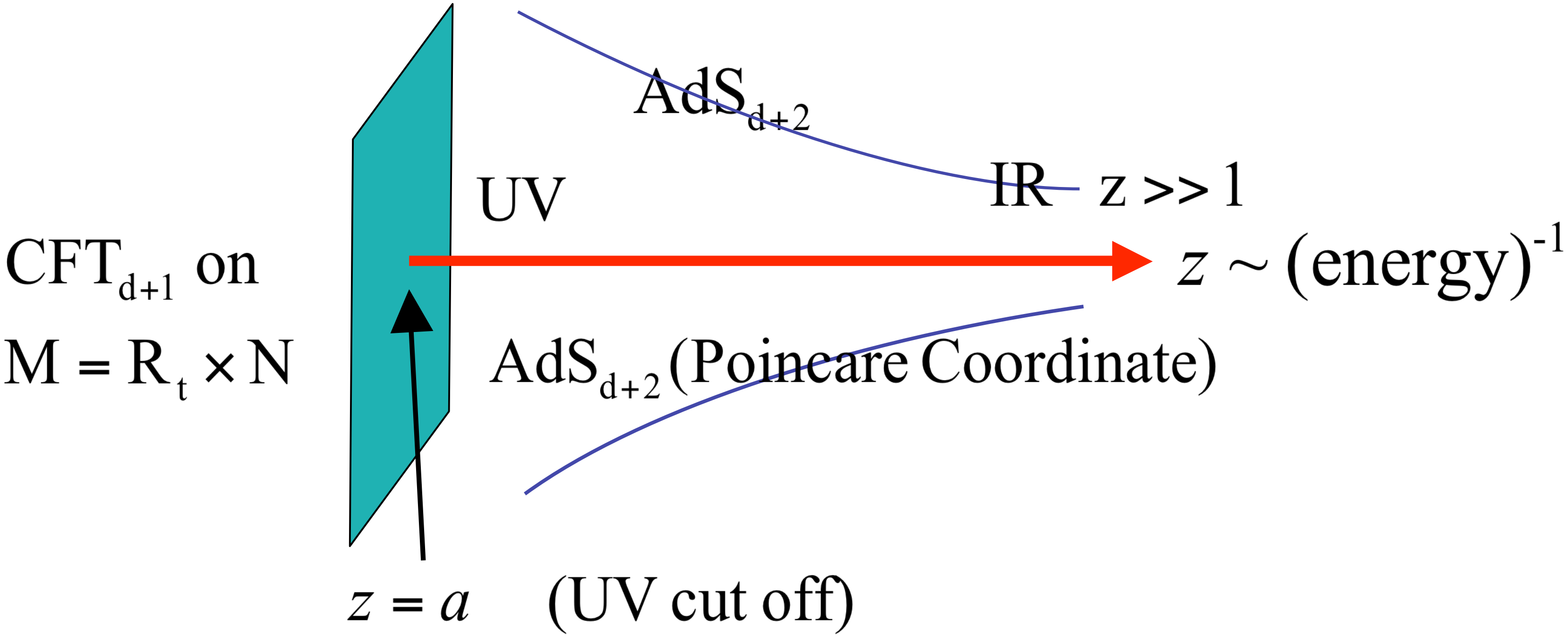
(ii) E.E. is proportional to the degrees of freedom.
It is non-vanishing even at zero temperature.
(~ a generalization of the central charge)

# Contents

# ② Holographic Entanglement Entropy Formula

Consider the AdS/CFT in Poincare Coordinate.

$$ds^2 = R^2 \frac{dz^2 - dx_0^2 + \Sigma_{i=1}^{d-1} dx_i^2}{z^2}.$$



## Our Holographic Formula

(1) Divide the space N is into A and B.

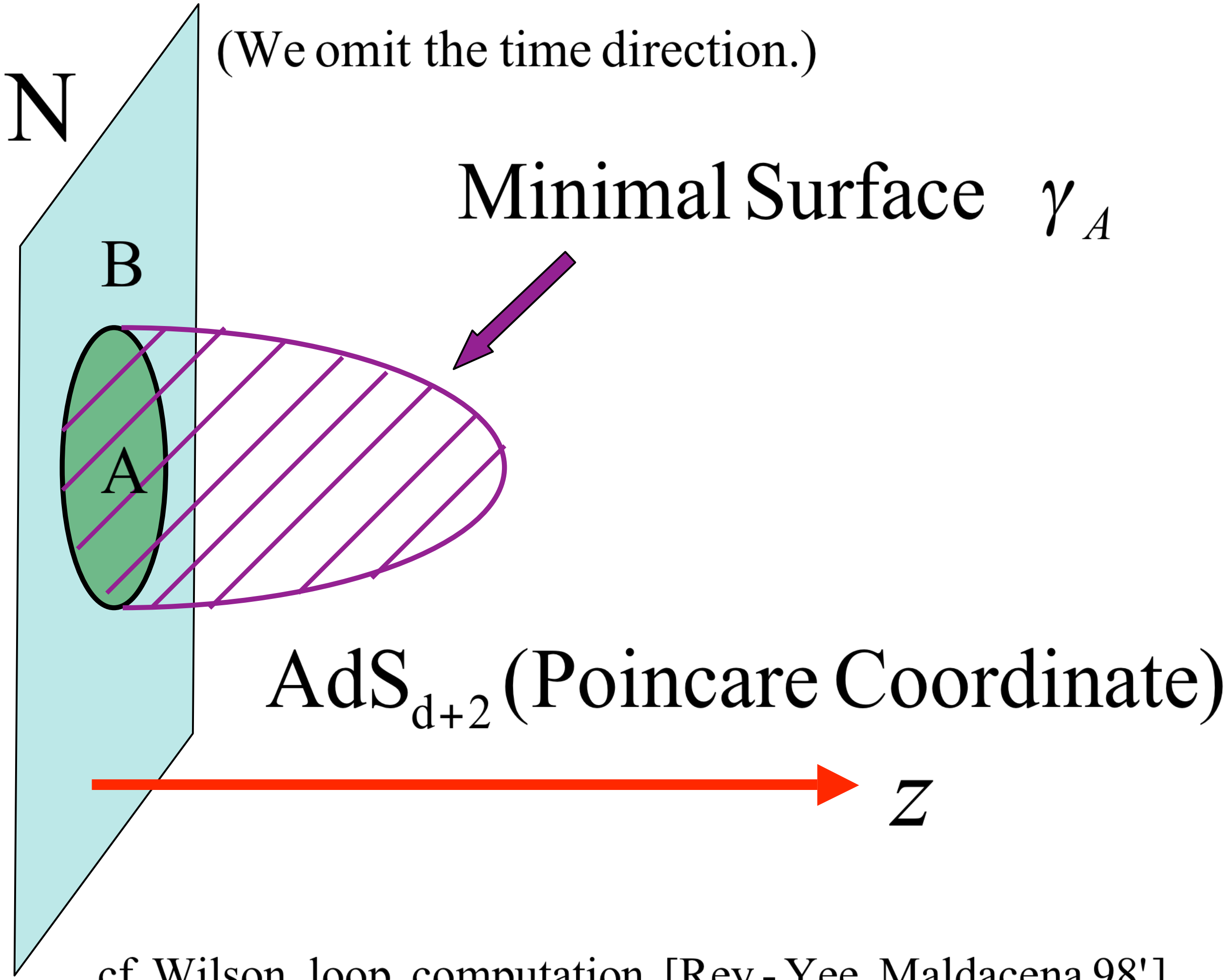
(2) Extend their boundary $\partial \mathrm{A}$ to the entire AdS space.
This defines a d dimensional surface.

(3) Pick up a minimal area surface and call this $\gamma_{\mathrm{A}}$.

(4) The E.E. is given by naively applying the
Bekenstein-Hawking formula

$$\boxed{S_A = \frac{\mathrm{Area}(\gamma_{\mathrm{A}})}{4G_N^{(d+2)}}.}$$

as if $\gamma_{\mathrm{A}}$ were an event horizon.



(We omit the time direction.)

Minimal Surface $\gamma_A$

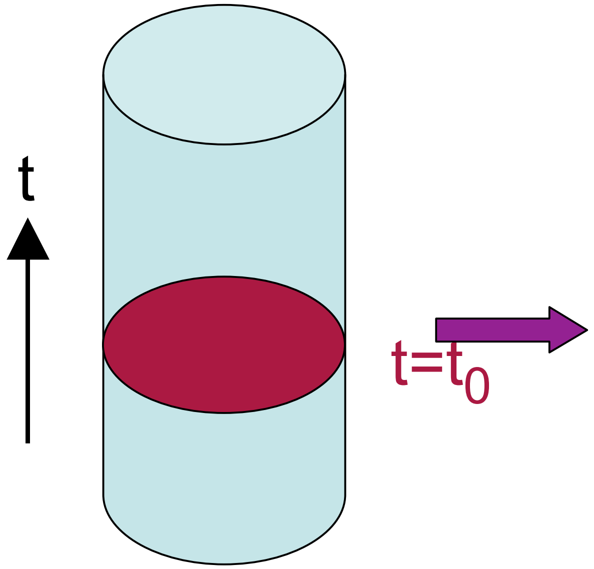
AdS$_{d+2}$ (Poincare Coordinate)

cf. Wilson loop computation [Rey - Yee, Maldacena 98']

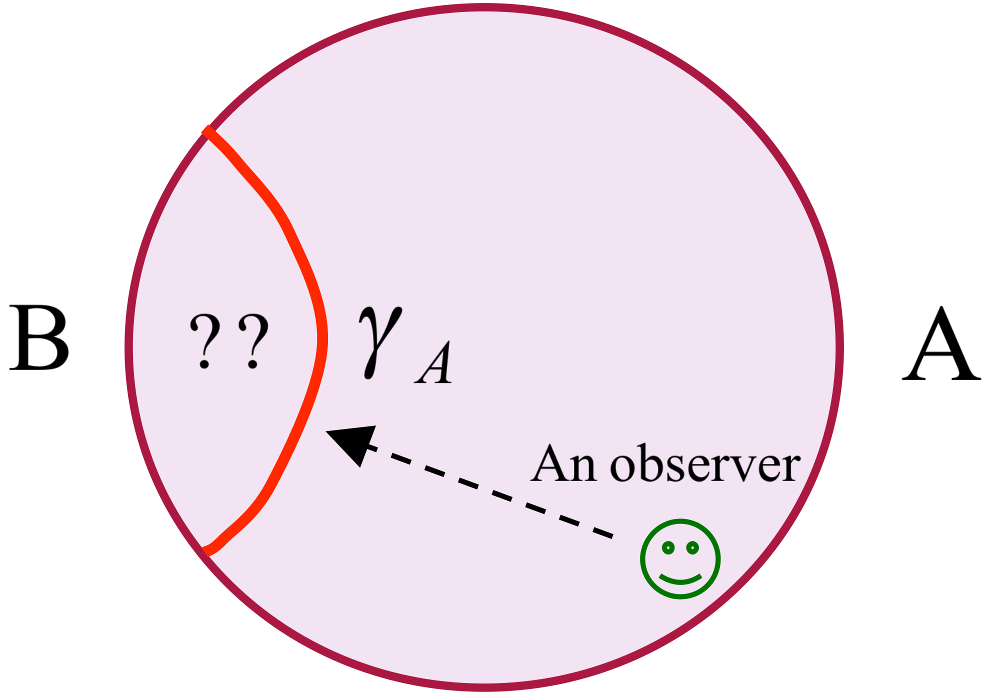
→ E.E. in 3D CFT ≅ Wilson Loop in 4D Gauge theory ?

## Motivation of this formula

Here we employ the global coordinate of AdS space and take its time slice at $t=t_0$.



$\mathrm{AdS}_{d+2}$
in global Coordinate



$\gamma_A$ is the Holographic Screen!

## Leading divergence

For a generic choice of $\gamma_{\mathrm{A}}$ , a basic property of AdS gives

$$\mathrm{Area}(\gamma_{\mathrm{A}}) \sim R^d \cdot \frac{\mathrm{Area}(\partial\gamma_{\mathrm{A}})}{a^{d-1}} + (\text{subleading terms}),$$

where R is the AdS radius.

Because $\partial\gamma_{\mathrm{A}} = \partial\mathrm{A}$, we find

$$S_A \sim \frac{\mathrm{Area}(\partial\mathrm{A})}{a^{d-1}} + (\text{subleading terms}).$$

This agrees with the ‘area law’ of E.E. in QFTs.

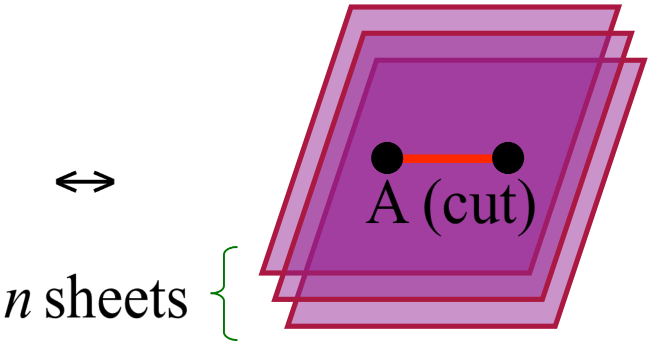
Bombelli-Koul-Lee-Sorkin 86’ , Srednicki 93’

## A proof of the holographic formula via GKP-Witten relation

[Fursaev hep-th/0606184]

In the CFT side, the (negative) deficit angle $2\pi(1-n)$ is localized on $\partial A$ .

$$\mathrm{Tr}_A[\rho_A^n] \quad \leftrightarrow$$



Naturally, it can be extended inside the bulk AdS by solving Einstein equation. We call $\gamma_{\mathrm{A}}$ this extended surface.

Let us apply the GKP-Witten formula $Z_{CFT} = e^{-S_{Gravity}(\phi_i)}$ in this background with the deficit angle $\Delta\varphi = 2\pi(1-n)$ .

The curvature is delta functionally localized on the deficit angle surface:

$$R = 4\pi(n-1)\cdot\delta(\gamma_A) + \text{regular terms.}$$

$$\Downarrow$$

$$S_{gravity} = \frac{1}{16\pi G_N}\int dx^{d+2}\sqrt{g}R + \ldots \quad \to \quad \frac{\mathrm{Area}(\gamma_{\mathrm{A}})}{4G_N}\cdot(n-1).$$

$$\Downarrow$$

$$S_A = -\frac{\partial}{\partial n}\log \mathrm{tr}_{\mathrm{A}}\rho^n = -\frac{\partial}{\partial n}\log\left(\frac{\mathrm{Z}_{\mathrm{n}}}{(\mathrm{Z}_1)^n}\right) = \frac{\mathrm{Area}(\gamma_{\mathrm{A}})}{4G_N}.$$

$$\delta S_{gravity} = 0 \quad \to \quad \gamma_A = \text{minimal surface!}$$

# ③ Several Examples

## (3-1) AdS3/CFT2 Case

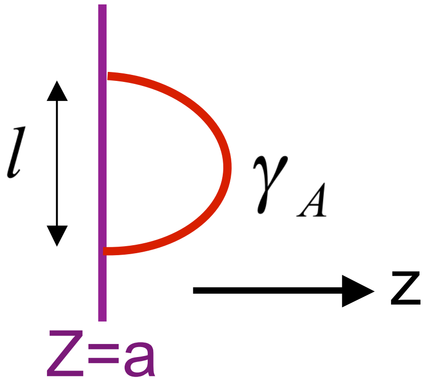
For any 2D CFTs, we can exactly compute E.E. for arbitrary choices of the subsystem A. [J.Cardy 2004']

In the dual gravity side, the minimal surfaces are now equivalent to the space-like geodesics.

Thus we can compare both results directly.
We checked that they completely agree with each other.

Consider the setup $\left\{\begin{array}{l}\text{total system = An infinite line}\\ \text{subsystem A = An interval of length } l\end{array}\right.$

The holographic formula leads to



$$S_A = \frac{|\gamma_A|}{4G_N^{(3)}} = \frac{c}{3}\log\left(\frac{l}{a}\right),$$

where we employed the celebrated expression $c = \frac{3R}{2G_N^{(3)}}.$ of the central charge.

This result from AdS3 perfectly agrees with the known formula in 2D CFT. (remember a is the UV cutoff)

## At Finite temperature T=1/β

In this case, the dual gravity background is the BTZ black Hole and the geodesic distance can be found exactly.

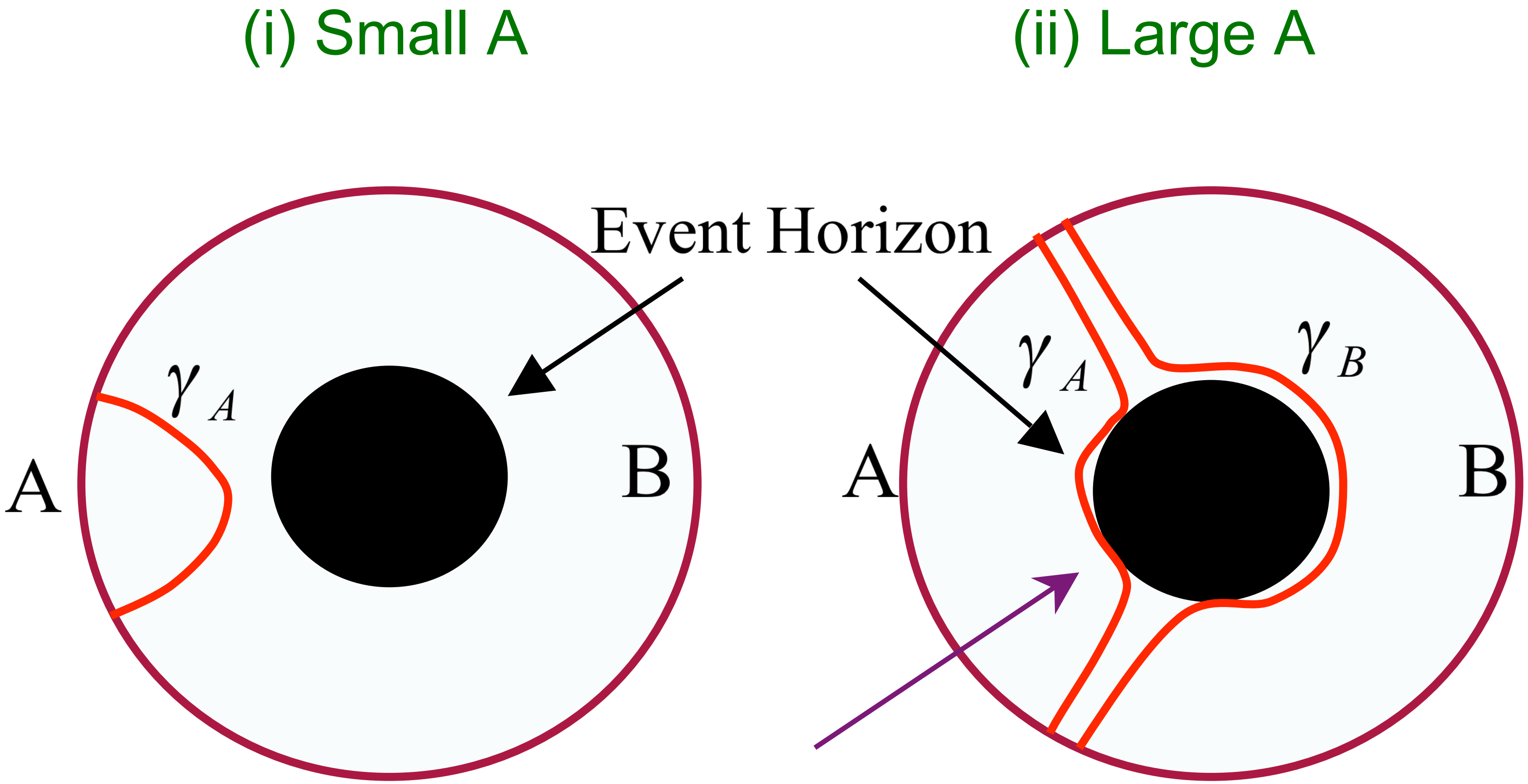
This again reproduces the known formula at finite temperature.

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right) \quad .$$

At higher temperature, it includes the thermal contribution

$$S_A \approx \frac{\pi}{3} c\, lT \qquad .$$

# Geometric Interpretation



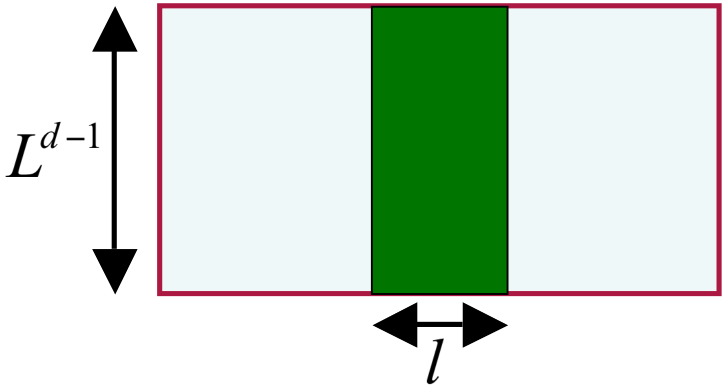
(i) Small A

(ii) Large A

When A is large (i.e. high temperature), $\gamma_A$ wraps a part of horizon. This leads to the thermal contribution $S_A \approx (\pi / 3) c\, l T$ to the entanglement entropy.

## (3-2) Higher Dimensional Case

Now we compute the holographic E.E. in the Poincare metric dual to a CFT on $R^{1,d}$. To obtain analytical results, we assume that the subsystem A = an infinite strip.



## Entanglement Entropy in N=4 Super Yang-Mills

Consider the setup of type IIB string on $\mathrm{AdS}_5 \times \mathrm{S}^5$,
which is dual to 4D N=4 SU(N) super Yang-Mills theory.

In this case, the holographic formula leads to
(this corresponds to the strongly coupled Yang-Mills)

$$S_A^{AdS} = \underbrace{\frac{N^2L^2}{2\pi a^2}}_{\text{Area law divergence}} - 2\sqrt{\pi}\underbrace{\left(\frac{\Gamma(2/3)}{\Gamma(1/6)}\right)^3 \frac{N^2L^2}{l^2}}_{\textbf{Finite and Universal !}}.$$

Now, we would like to compare this with free Yang-Mills result.

The AdS results numerically reads

$$S_A^{AdS} = K' \cdot \frac{N^2 L^2}{a^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}.$$

On the other hand, the free Yang-Mills result becomes

$$S_A^{freeCFT} = K \cdot \frac{N^2 L^2}{a^2} - 0.078 \cdot \frac{N^2 L^2}{l^2}.$$

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills. Thus this is regarded as a semi-quantitative matching.

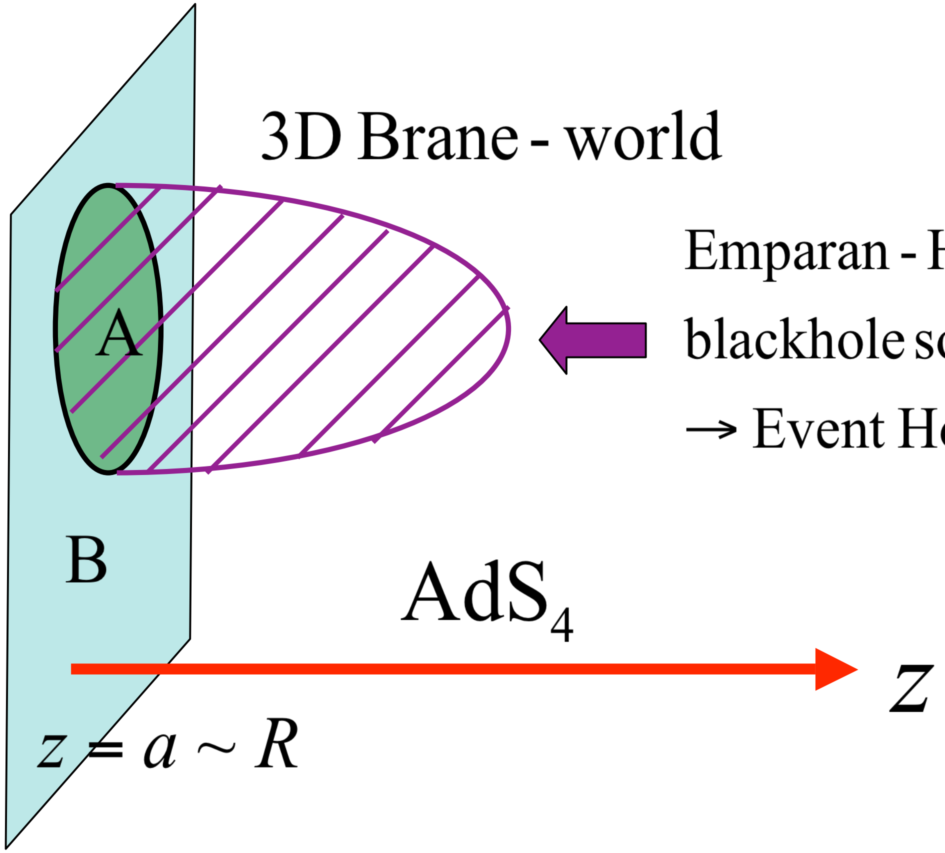
[cf. **4/3 problem** in thermal entropy, Gubser-Klebanov-Peet 96']

# (3-3) Black hole entropy and Entanglement Entropy

[Emparan hep-th/0603081]

**Claim**

Entropy of 3D quantum black hole = Entanglement Entropy



Emparan - Horowitz - Myers

blackhole solution (d = 3)

→ Event Horizon : $\gamma_A$

## (3-4) Non-supersymmetric AdS bubble backgrounds

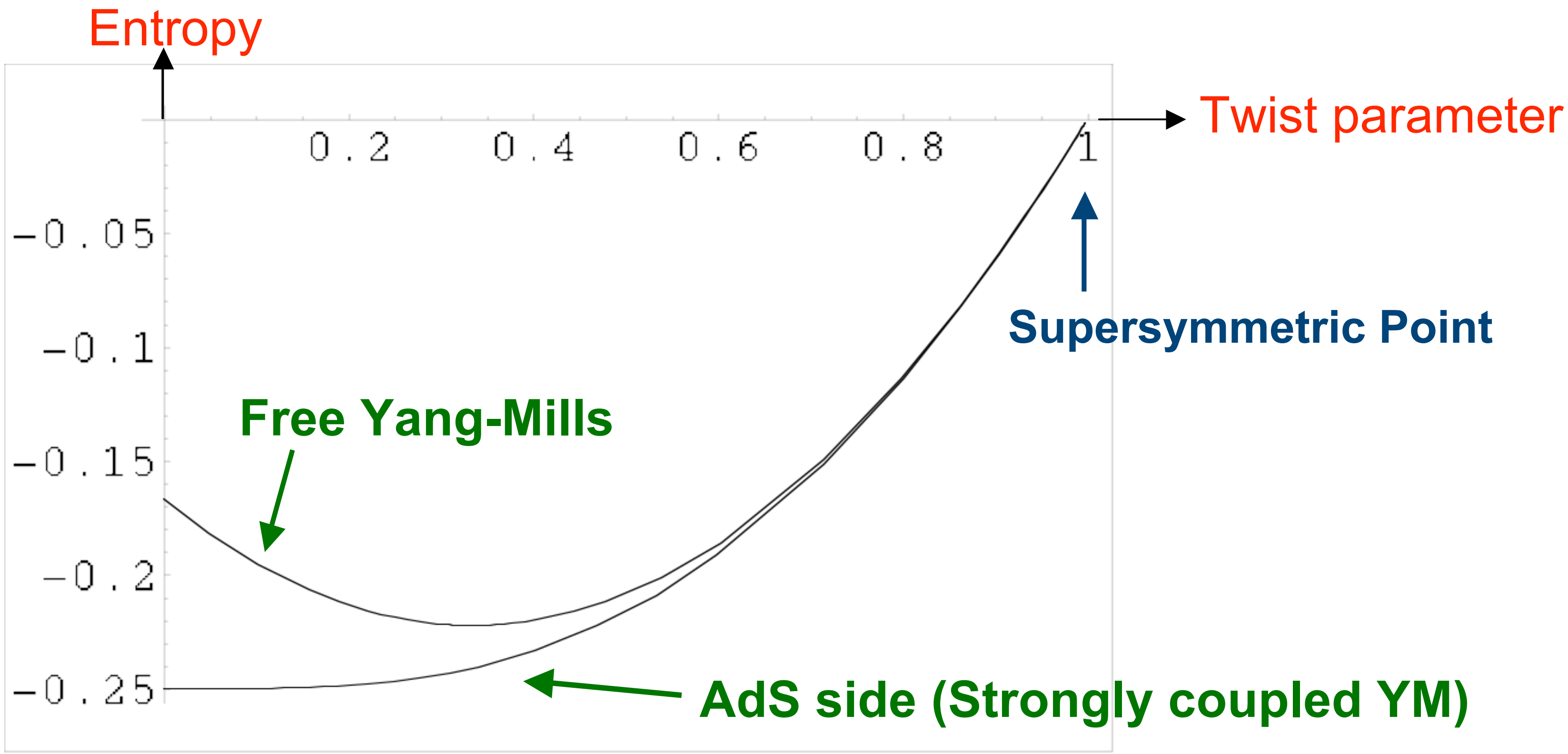
As a final example we consider the asymptotic AdS backgrounds which are obtained from the double Wick rotation of R-charged AdS black holes.

They are a series of static bubble solutions dual to the N=4 SYM with SUSY breaking twisted boundary conditions.

At a specific value of the twist parameter, the N=4 SUSY is recovered.

→ We can break the SUSY very softly!

The entanglement entropies computed in the free Yang-Mills and the AdS gravity agree nicely!



This is a quantitative evidence for AdS/CFT in non-SUSY b.g.!

# ④ Conclusions

- We have given a holographic interpretation of entanglement entropy via AdS/CFT duality.

**Generalized Area Law**

Minimal Surface Area in AdS
= Entanglement Entropy in CFT

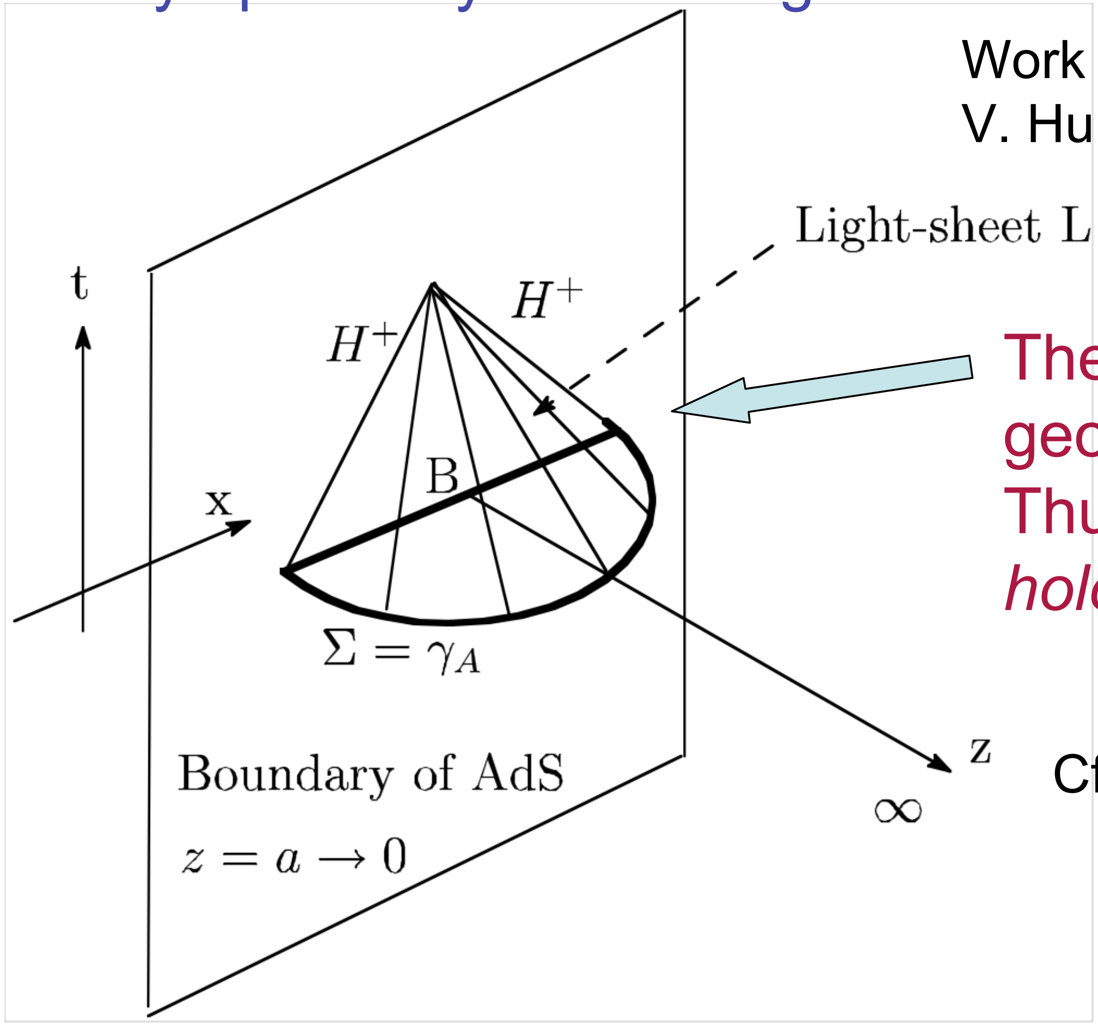
This strongly suggests

Minimal Surface $\gamma_A$ in AdS = Holographic Screen for A

It clarifies the issue of *the locality in AdS/CFT*:

*Which place in AdS encodes the information of the dual CFT restricted to a given region??*

## This area law will be true even in time-dependent asymptotically AdS backgrounds.



Work in progress with
V. Hubeny and M. Rangamani

The expansion of the null geodesic is vanishing. Thus it is the *preferred holographic screen*.

Cf. Bousso Bound

*AdS3 in Poincare Coodinate*

# Happy Birthday to Yoneya-san !