

Moduli Space of Domain Wall Junctions and Loops : On the occasion of Yoneya-san's 60th birthday

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“Webs of Domain Walls in Supersymmetric Gauge Theories”, Phys.Rev.**D72** (2005) 085004,

“ Non-Abelian Webs of Walls”, Phys.Lett. **B632** (2006) 384,

“ Effective Action of Wall Loops ”, Phys.Rev.[hep-th/0612003]

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1 Introduction

Domain walls \sim D-branes : **1/2 BPS**

Junctions of walls \sim **Junctions** of branes

(p, q) string $((p, q)$ 5-brane) junction in IIB string: **1/4 BPS**

Wall junctions as 1/4 BPS states in SUSY field theories.

Domain walls are expected to make a **network-like webs**

Our purpose: To construct **all solutions** of wall webs as 1/4 BPS states in

$d = 4$, $\mathcal{N} = 2$ SUSY $U(N_C)$ gauge theories with $N_F (> N_C)$

hypermultiplets in the fundamental representation with complex masses

Results:

1. **Webs of Domain Walls** are constructed as 1/4 BPS states in $\mathcal{N} = 2$ SUSY $U(N_C)$ **Non-Abelian Gauge Theories** in 4 dimensions with N_F hypermultiplets in the fundamental representation.
2. **Exact Solutions** of Webs of Walls are obtained for $g^2 \rightarrow \infty$.
3. Normalizable moduli of web of walls = **loops of walls**
4. **Metric** of a single triangle loop of walls is explicitly worked out and can be understood as **kinetic energy of walls and junctions**.

SUSY $U(N_C)$ Gauge Theory with N_F Flavors

$\mathcal{N} = 2$ SUSY in 3+1 dim. $\mu, \nu = 0, 1, 2, 3$, $\alpha, \beta = 1, 2$

Vector multiplets : W_μ Gauge field, Σ_α 2 Real Scalars ($N_C \times N_C$ matrix)

Gauge coupling g for $U(N_C)$, **Fayet-Iliopoulos** (FI) parameter c

Hypermultiplets : $(H^i)^{rA} \equiv H^{irA}$ Complex Scalar ($N_C \times N_F$ matrix)

($i = 1, 2$; Color $r = 1, \dots, N_C$; Flavor $A = 1, \dots, N_F$)

Hypermultiplet Masses $(M_1)^A_B \equiv m_A \delta^A_B$, $(M_2)^A_B \equiv n_A \delta^A_B$

Non-degenerate masses: $m_A + in_A \neq m_B + in_B$

Minimal kinetic terms

$$\mathcal{L} = \text{Tr} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{g^2} \sum_{\alpha=1}^2 \mathcal{D}_\mu \Sigma_\alpha \mathcal{D}^\mu \Sigma_\alpha + \mathcal{D}_\mu H^i (\mathcal{D}^\mu H^i)^\dagger \right] - V$$

$$V = \text{Tr} \left[\sum_{\alpha=1}^2 (H^i M_\alpha - \Sigma_\alpha H^i) (H^i M_\alpha - \Sigma_\alpha H^i)^\dagger - \frac{1}{g^2} [\Sigma_1, \Sigma_2]^2 \right]$$

$$+ \text{Tr} \left[(Y^3)^2 + g^2 H^2 H^{1\dagger} H^1 H^{2\dagger} \right],$$

$$Y^3 \equiv \frac{g^2}{2} (H^1 H^{1\dagger} - H^2 H^{2\dagger} - c 1_{N_C})$$

Color-flavor locking vacua $\langle A_1 A_2 \cdots A_{N_C} \rangle$

$$H^{1rA} = \sqrt{c} \delta^{Ar}_A, \quad H^{2rA} = 0$$

$$\Sigma \equiv \Sigma_1 + i\Sigma_2 = \text{diag} \left(m_{A_1} + in_{A_1}, m_{A_2} + in_{A_2}, \cdots, m_{A_{N_C}} + in_{A_{N_C}} \right)$$

$N_F C_{N_C} = N_F! / (N_C! (N_F - N_C)!)$ discrete SUSY vacua

Higgs Phase : Walls, Vortices are the only **elementary** solitons

Instantons, monopoles, junctions are **composite** solitons

2 1/4 BPS Equations

Dependence on x^1, x^2 , 2 D Poincaré invariance $\rightarrow W_{0,3} = 0$

Bogomol'nyi completion of Energy density (assuming $H^2 = 0, H^1 \equiv H$)

$$\begin{aligned} \mathcal{E} = & \text{Tr} \left[\frac{1}{g^2} (F_{12} - i[\Sigma_1, \Sigma_2])^2 + \frac{1}{g^2} (\mathcal{D}_1 \Sigma_2 - \mathcal{D}_2 \Sigma_1)^2 \right. \\ & + \sum_{\alpha=1,2} (\mathcal{D}_\alpha H - HM_\alpha + \Sigma_\alpha H) (\mathcal{D}_\alpha H - HM_\alpha + \Sigma_\alpha H)^\dagger \\ & \left. + \frac{1}{g^2} (\mathcal{D}_1 \Sigma_1 + \mathcal{D}_2 \Sigma_2 - Y^3)^2 \right] + \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Y} + \sum_{\alpha=1,2} \partial_\alpha J_\alpha \end{aligned}$$

$$\mathcal{Z}_1 \equiv c\partial_1 \text{Tr}\Sigma_1, \quad \mathcal{Z}_2 \equiv c\partial_2 \text{Tr}\Sigma_2, \quad \mathcal{Y} \equiv \frac{2}{g^2} \partial_\alpha \text{Tr} (\epsilon^{\alpha\beta} \Sigma_2 \mathcal{D}_\beta \Sigma_1)$$

1/4 BPS equations for domain wall webs

$$\begin{aligned} F_{12} &= i [\Sigma_1, \Sigma_2], \quad \mathcal{D}_1 \Sigma_2 = \mathcal{D}_2 \Sigma_1 \\ (\mathcal{D}_1 + \Sigma_1)H &= HM_1, \quad (\mathcal{D}_2 + \Sigma_2)H = HM_2 \\ \mathcal{D}_1 \Sigma_1 + \mathcal{D}_2 \Sigma_2 &= Y^3 \end{aligned}$$

Integrability condition

$$\begin{aligned} F_{12} = i [\Sigma_1, \Sigma_2], \quad \mathcal{D}_1 \Sigma_2 = \mathcal{D}_2 \Sigma_1 &\rightarrow [\mathcal{D}_1 + \Sigma_1, \mathcal{D}_2 + \Sigma_2] = 0 \\ \rightarrow N_C \times N_C \text{ non-singular matrix } S(x^\alpha) &\text{ as simultaneous solution} \end{aligned}$$

$$W_1 - i\Sigma_1 = -iS^{-1}\partial_1 S, \quad W_2 - i\Sigma_2 = -iS^{-1}\partial_2 S$$

Hypermultiplet BPS equations are solved by $S(x^\alpha)$ as

$$H = S^{-1} H_0 e^{M_1 x^1 + M_2 x^2}$$

Moduli matrix H_0 : $N_C \times N_F$ constant complex matrix of rank N_C

Master equation in terms of a **gauge invariant** matrix $\Omega \equiv S S^\dagger$

$$\frac{1}{cg^2} [\partial_1 (\partial_1 \Omega \Omega^{-1}) + \partial_2 (\partial_2 \Omega \Omega^{-1})] = \mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}$$

$$\Omega_0 \equiv c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger$$

Moduli matrix H_0 **contains all moduli parameters**

(H_0, S) and (H'_0, S') give the same configurations, if related by

$$H_0 \rightarrow H'_0 = V H_0, \quad S \rightarrow S' = V S, \quad V \in GL(N_C, \mathbb{C})$$

Independent moduli are equivalence class defined by $(H_0, S) \sim (H'_0, S')$

The **total moduli space**: the **complex Grassmann manifold**

$$\mathcal{M}_{\text{tot}}^{\text{webs}} \simeq G_{N_F, N_C} = \{H_0 \mid H_0 \sim V H_0, V \in GL(N_C, \mathbb{C})\}$$

Existence and uniqueness of solutions of the master equation to be proved

Exact Solution at $g \rightarrow \infty$: NLSM

Strong coupling limit $g^2 c / \Delta m \gg 1$: BPS Eq. for $\Omega \rightarrow$ Algebraic equation

$$\Omega^{g \rightarrow \infty} = \Omega_0 = c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger$$

Abelian gauge theory ($N_C = 1$): configurations of scalar fields are

$$H^A = \sqrt{c} \frac{H_0^A e^{m_A x^1 + n_A x^2}}{\sqrt{\sum_{B=1}^{N_F} |H_0^B|^2 e^{2(m_B x^1 + n_B x^2)}}}$$

3 Webs of Walls

Take $U(1)$ gauge theory as a simple example

Moduli matrix: $H_0 = \sqrt{c}(e^{a_1+ib_1}, \dots, e^{a_{N_F}+ib_{N_F}})$

$\log \Omega \sim \log \Omega_0$ outside the core of the wall

Position of the domain wall: equal weights of the vacua i, j

$$(m_i - m_j)x^1 + (n_i - n_j)x^2 + a_i - a_j = 0$$

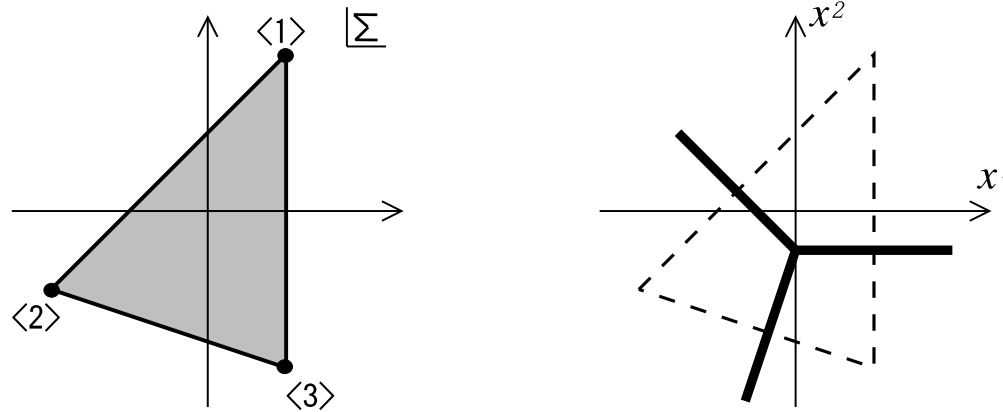


Figure 1: The minimal model for 3-pronged wall junction. The left one is the **grid diagram** in the complex Σ plane and the right one is the **web diagram** in the configuration space.

Wall Junction

$N_F = 3$ with 3 discrete vacua labeled by $\langle A \rangle$ ($A = 1, 2, 3$)

1/4 BPS wall junction: a triangle with 3 vertices at $m_A + in_A$ in Σ

Polygons in the Σ plane = *grid diagrams* ((p, q) string/5-brane webs)

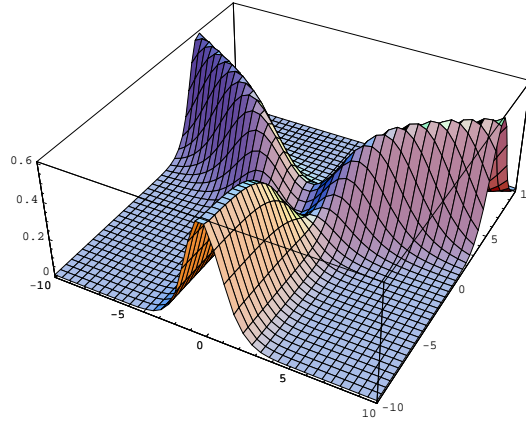


Figure 2: Binding energy at the junction point: The energy density is numerically evaluated for the moduli matrix $H_0 e^{m \cdot x} = \left(e^{x^2}, e^{\sqrt{3}x^1/2 - x^2/2}, e^{-\sqrt{3}x^1/2 - x^2/2} \right)$, gauge coupling $g = 1$ and FI parameter $c = 1$.

Boundary conditions for 1/4 BPS webs: \rightarrow walls at $x^1, x^2 \rightarrow \infty$

Walls with tension $T^{AB} = (Z_2^{AB}, -Z_1^{AB})$ pull junction along the wall

Central charge: $(Z_1^{AB}, Z_2^{AB}) \equiv c(m_B - m_A, n_B - n_A)$

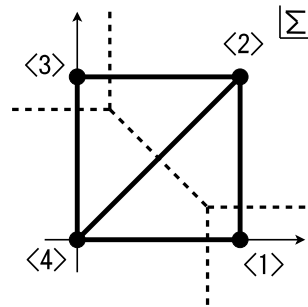
Junction configuration: **web diagram** dual to the **grid diagram**

Edges of grid diagram $\leftrightarrow \mathbf{x}^1, \mathbf{x}^2 \rightarrow \infty$ (Walls)

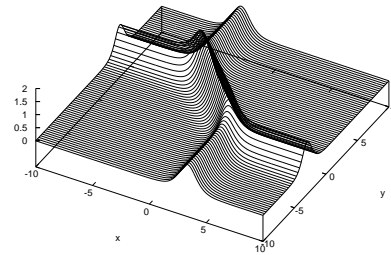
Area of triangle \leftrightarrow junction charge \mathbf{Y} : contributes to the energy

negatively ($\mathbf{Y} < \mathbf{0}$) in $U(1)$ gauge theories (**binding energy**)

positively ($\mathbf{Y} > \mathbf{0}$) in **nonAbelian junctions**



(a) grid diagram



(b) energy density ($g^2 \rightarrow \infty$)

Figure 3: Wall web with 4 external legs of walls. Grid diagram:(a), and energy density:(b).
 ($[m_A, n_A] = \{[1, 0], [1, 1], [0, 1], [0, 0]\}$)

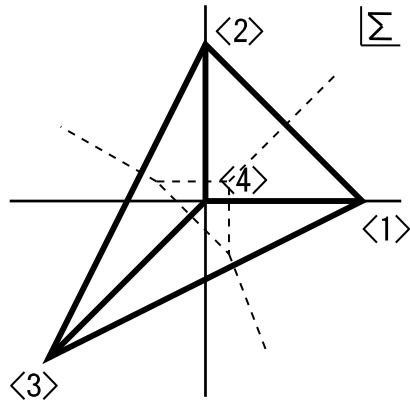
Web of Walls

$U(1)$ gauge theories with N_F flavors (Web diagrams with N_F faces)

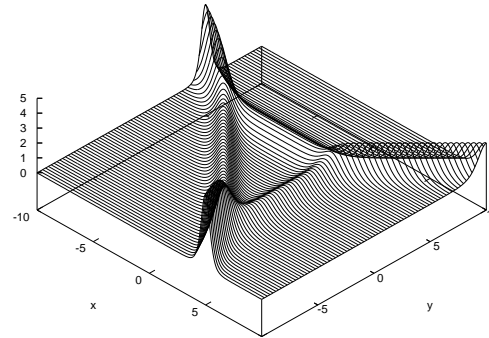
Two kinds of web diagrams: **tree** diagram, and diagram with **loops**

$N_F = 4$ model: Moduli matrix: $H_0 = \sqrt{c} (e^{a_1+ib_1}, e^{a_2+ib_2}, e^{a_3+ib_3}, e^{a_4+ib_4})$

Homogeneous coordinates of the total moduli space CP^3



(a) grid diagram



(b) energy density ($g^2 \rightarrow \infty$)

Figure 4: Web with 1 loop in the $N_F = 4$ model. Grid diagram:(a), and energy density:(b).
 $([m_A, n_A] = [1, 0], [0, 1], [-1, -1], [0, 0])$

Wall Loops

Different choices of mass in the $N_F = 4$ model \rightarrow wall web with a **loop**

4 Effective Action of Wall Loops

Size of the Loop (and associated phase) is the **normalizable moduli**

$$H_0 = \sqrt{c}(1, 1, 1, \phi) \quad \text{with} \quad \phi = e^{r+i\theta}$$

$$\mathcal{L}^{eff} = K_{ij^*}(\phi, \phi^*) \partial^\mu \phi^i \partial_\mu \phi^{j^*}, \quad K(\phi, \phi^*) = K_w(\phi, \phi^*) + K_g(\phi, \phi^*)$$

$$K_w(\phi, \phi^*) \equiv \int d^2x \, c \log \det \Omega, \quad K_g(\phi, \phi^*) \equiv \int d^2x \, \frac{1}{2g^2} \text{Tr}(\Omega^{-1} \partial_\alpha \Omega)^2$$

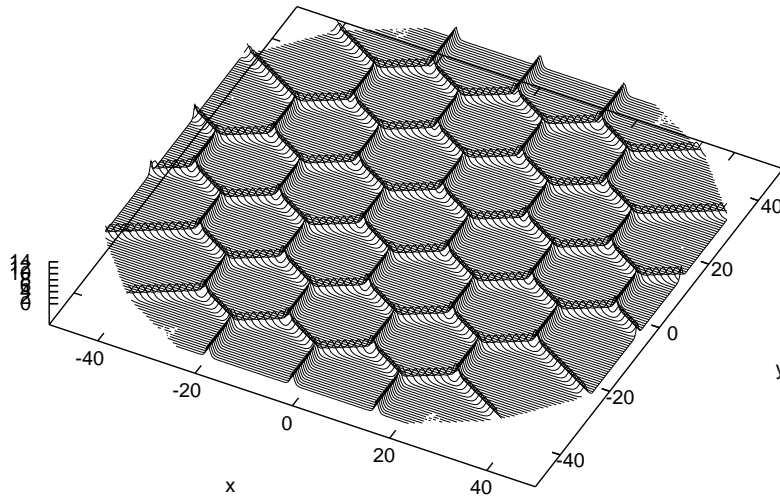


Figure 5: Honeycomb web of domain walls which divides 37 vacua (\mathbf{CP}^{36}) with 18 external walls and 19 internal loops of walls.

Take $U(1)$ gauge theory with $N_{N_F} = 4$, $[m_4, n_4] = [0, 0]$

Metric at Small Loops in strong coupling limit $g^2 \rightarrow \infty$

$$K_w \equiv c \int d^2x \left[\log \Omega_0 - \log \tilde{\Omega}_0 \right] = c \int d^2x \log \left(1 + \frac{|\phi|^2}{\tilde{\Omega}_0} \right)$$

$$\Omega_0 = e^{2m_1 \cdot x} + e^{2m_2 \cdot x} + e^{2m_3 \cdot x} + |\phi|^2$$

$$\tilde{\Omega}_0 \equiv e^{2m_1 \cdot x} + e^{2m_2 \cdot x} + e^{2m_3 \cdot x}, \quad m_A \cdot x \equiv m_A x^1 + n_A x^2$$

Areas of triangles in field space $\Delta_{[123]}$

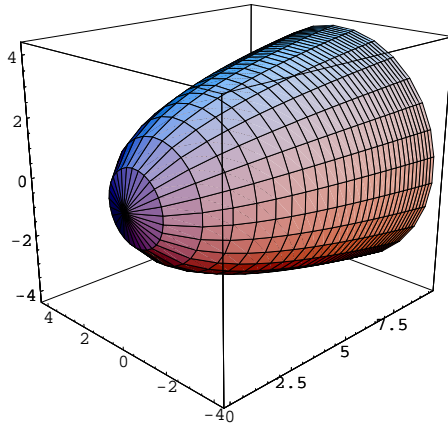


Figure 6: The moduli space of single triangle loop around $\phi = \mathbf{0}$ where the loop shrinks. $U(1)$ isometry is the phase modulus. The other direction is the size modulus of the loop.

For $|\phi|^2 \leq \exp(-\sum \alpha_i \log \alpha_i)$, ($\alpha_i \equiv (m_j \times m_k) / \Delta_{[123]}$)

$$K_w = \frac{c}{4\Delta_{[123]}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \Gamma(\alpha_1 n) \Gamma(\alpha_2 n) \Gamma(\alpha_3 n)}{n \Gamma(n)} |\phi|^{2n}$$

Scalar curvature is finite (**nonsingular**) even at $\phi = \mathbf{0}$ (vanishing loop)

$$R = \frac{16\Delta_{[123]} \Gamma(2\alpha_1) \Gamma(2\alpha_2) \Gamma(2\alpha_3)}{c (\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3))^2} + \mathcal{O}(|\phi|^2)$$

Metric at Large Loops (arbitrary gauge coupling g)

Tropical limit: Retaining the largest term in each region

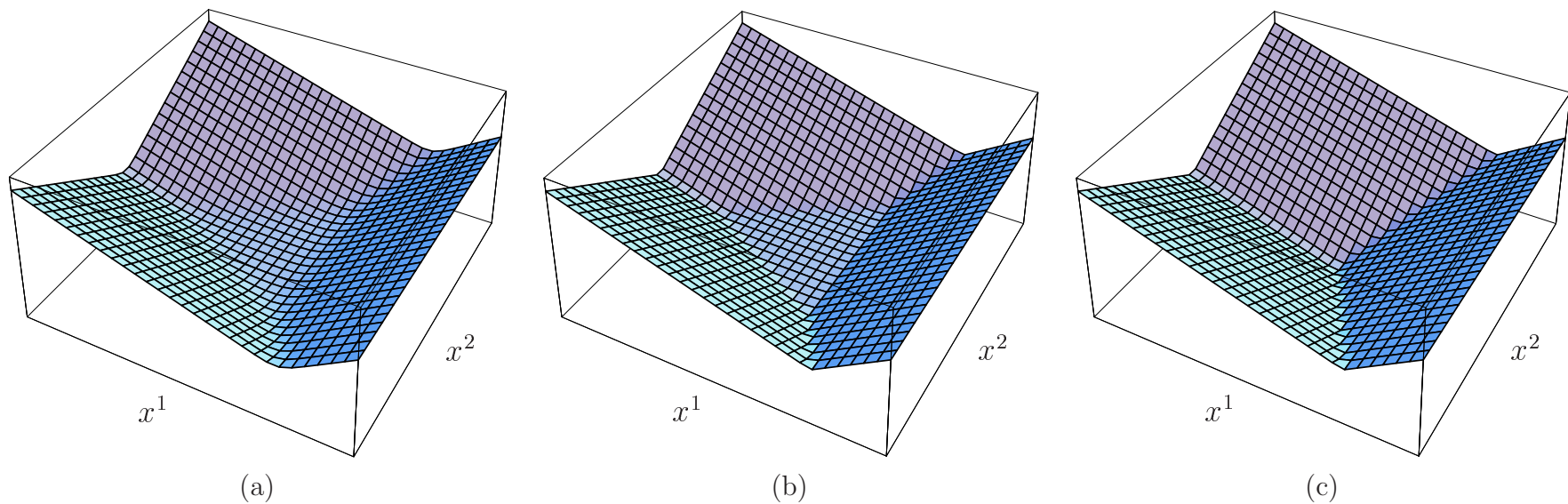


Figure 7: (a) plot of $\log \Omega_0$, (b) tropical limit of $\log \Omega_0$, (c) tropical limit of $\log \tilde{\Omega}_0$

K_w is given by c times volume of the tetrahedron

$$K_w^{trop} = \frac{c}{24\Delta_{[123]}} \frac{1}{\alpha_1\alpha_2\alpha_3} (\log |\phi|^2)^3$$

Total Kähler metric

$$ds^2 = \frac{c}{\Delta_{[123]}} \left[\frac{r}{\alpha_1\alpha_2\alpha_3} - \frac{1}{g^2c} \left(\frac{|m_{12}|^2}{\alpha_3} + \frac{|m_{23}|^2}{\alpha_1} + \frac{|m_{31}|^2}{\alpha_2} \right) \right] (m^2 dr^2 + d\theta^2)$$

Kinetic energies of walls and junctions due to the moduli motion

5 Conclusion

1. **Webs of Domain Walls** are constructed as $1/4$ BPS states in $\mathcal{N} = 2$ SUSY $U(N_C)$ **Non-Abelian Gauge Theories** in 4 dimensions with N_F hypermultiplets in the fundamental representation.

2. **Total moduli space** of the webs of walls is given by a **complex Grassmann manifold** described by the **moduli matrix** H_0

$$G_{N_F, N_C} \simeq SU(N_F) / [SU(N_F - N_C) \times SU(N_C) \times U(1)]$$

3. **Exact Solutions** of Webs of Walls are obtained for $g^2 \rightarrow \infty$.

4. A General Formula for the **Effective Lagrangian** is obtained.

5. **Abelian junction** has negative junction charge (binding energy). **Non-Abelian Junction** has positive junction charge (Hitchin system).

6. Normalizable moduli of web of walls = **loops of walls**

7. **Metric** of a single triangle loop of walls is explicitly worked out and can be understood as **kinetic energy of walls and junctions**.