Emergence of Space-time in Matrix Model

Komaba 07

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Abstract

IIB matrix model is a candidate for the constructive definition of string theory. It is nothing but the large-N reduced model of 10D super Yang-Mills theory

$$S = -\frac{1}{g^2} Tr(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\Gamma^\mu[A^\mu, \Psi])$$

IKKT

Jevicki-Yoneya

A^μ ($\mu = 1 \sim 10$),

Ψ (10D Majorana-Weyl) : N×N hermitian.

But this theory seems to describe fluctuations around a flat space-time.

The basic question is whether it can describe curved space-times, and it really contains invariance of the general relativity, diffeomophism and local Lorentz invariance, or not.

Here, we will show that indeed it is the case, if we introduce a *new interpretation* for the matrices.

A_μ can be regarded as momentum.

In the original interpretation, A_μ are the space-time coordinates.

$$S_{Schild} = \int d^2\xi \sqrt{g} \left\{ X^\mu, X^\nu \right\}^2 + \cdots$$

$$\downarrow \quad \begin{array}{ccc} \int d^2\xi\sqrt{g} & \rightarrow & Tr \\ \{\, , \,\} & \rightarrow & i[\, , \,] \\ X^\mu & \rightarrow & A_\mu \end{array}$$

$$S_{IIB} = -Tr\left[A_\mu, A_\nu\right]^2 + \cdots$$

emission vertex string state

$$\int d^2\xi \sqrt{g}\, \exp\left(i\, k_\mu X_\mu(\xi)\right) \quad \leftarrow \quad \left|k_\mu(\sigma)\right\rangle = \exp\left(i \oint d\sigma\, k_\mu(\sigma) X^\mu(\sigma)\right)$$

$\downarrow$ mode expansion $\downarrow$

$$Tr\left(\exp\left(i\, k_\mu A_\mu\right)\right) \quad \leftarrow \quad \left|k_\mu(\sigma)\right\rangle = Tr\left(P\exp\left(i \oint d\sigma\, k_\mu(\sigma) A_\mu\right)\right)$$

It is natural to regard A_μ as space-time coordinates, and

$Tr\left(P\exp\left(i \oint d\sigma\, k_\mu(\sigma) A_\mu\right)\right)$ as emission of the string state $\left|k_\mu(\sigma)\right\rangle$.

If we introduce the T-dual picture $k_\mu(\sigma) \leftrightarrow x^{\mu\prime}(\sigma)$, this becomes

$$w\left[x^\mu(\sigma)\right] = Tr\left(P\exp\left(i \oint d\sigma\, x^{\mu\prime}(\sigma) A_\mu\right)\right),$$

which is nothing but Wilson loop in the large-N reduced model.

Parisi,
Gross, Kitazawa,
Bhanot, Heller, Neuberger,
Das, Wadia

quenched large-N reduced model

D-matrix model

$$S_R = -\frac{N}{4{g_R}^2} Tr\left[A_\mu, A_\nu\right]^2, \left(\mu = 1, \cdots, D\right)$$

Here we regard matrices as

$$A_\mu \in End(V) \ ,$$

where $V = C^\infty\left(\mathbb{R}^d\right)$.

(We introduce IR and UV cut off, s.t. *dim V=N.*)

Take a classical solution

$$A_\mu^{(0)} = \begin{cases} i\partial_\mu \ , & \left(\mu = 1, \cdots, d\right) \\ 0, & \left(\mu = d+1, \cdots, D\right) \end{cases},$$

and expand A_μ around it

$$A_\mu = A_\mu^{(0)} + \tilde{A}_\mu ,$$

with the condition of "quenching" imposed *by hand*

diagonal elements of $\tilde{A}_\mu = 0$ (in the mom. rep).

reduced model S_R	$\underset{N \to \infty}{\cong}$	*D*-dim Yang-Mills theory reduced to *d*-dimensions

$${g_R}^2 {\Lambda_{UV}}^{4-d} = {g_{YM}}^2 .$$

We have forced the diagonal elements to distribute uniformly by hand. But in IIB matrix model their dynamics is rather complicated.

Aoki, Iso, Kitazawa, Tada, HK

In fact the one-loop effective action for the diagonal elements

$$A_\mu = \begin{pmatrix} x_\mu{}^{(1)} & & * \\ & \ddots & \\ * & & x_\mu{}^{(N)} \end{pmatrix}, \quad \psi = \begin{pmatrix} \xi^{(1)} & & * \\ & \ddots & \\ * & & \xi^{(N)} \end{pmatrix}$$

is given by

$$S_{\mathrm{eff}}{}^{\text{1-loop}}(x,\xi) = \sum_{i<j} tr\left(\frac{S_{(i,j)}{}^4}{4} + \frac{S_{(i,j)}{}^8}{8} \right),$$

$$S_{(i,j)} = \left(\bar{\xi}^{(i)} - \bar{\xi}^{(j)}\right)\Gamma^{\mu\alpha\nu}\left(\xi^{(i)} - \xi^{(j)}\right)\frac{x_\alpha{}^{(i)} - x_\alpha{}^{(j)}}{\left(x_\beta{}^{(i)} - x_\beta{}^{(j)}\right)^2} \quad .$$

If $\xi^{(i)}$ were not there, $S_{\mathrm{eff}}{}^{\text{1-loop}}(x,\xi)$ would be zero, and the $x_\mu{}^{(i)}$ integration would be infinite.

In the real expression, the $\xi^{(i)}$ integration produces negative powers of $x_\mu{}^{(i)}$, which make the $x_\mu{}^{(i)}$ integration convergent.

Although we do not know how close it is to the real vacuum, if we consider fluctuations around a background, for example,

$$A_\mu^{(0)} = \begin{cases} i\partial_\mu \ , & (\mu = 1, \cdots, 4) \\ 0 \ , & (\mu = 5, \cdots, 10) \end{cases},$$

and quench the diagonal elements by hand, IIB matrix model becomes equivalent to the large-N $\mathcal{N}$=4 super Yang-Mills theory.

4D flat space-time or $AdS_5 \times S_5$	$\subset$	Path integral of IIB matrix model

Question:

We have seen that a set of d matrices

$$A_\mu = i\partial_\mu\ ,\ (\mu = 1,\cdots,d)$$

represents d-dimensional flat space.

Can any d-dimensional curved space be represented by a set of d matrices?

A naïve answer is to consider

$$A_a = i\nabla_a\ ,$$

where $\nabla_a\ (a = 1 \sim d)$ is the covariant derivative on the curved space that we want to represent.

However this does not work.

Difficulty

The covariant derivative $\nabla_a\ (a = 1 \sim d)$ maps a scalar field to a vector field, and a vector field to a tensor field, and so on. Therefore, it can *not* simply be regarded as a linear transformation on some space.

The difficulty may become clearer, if we compare the product $\nabla_a \nabla_b$ and $A_a A_b$. In $\nabla_a \nabla_b$, the spin connection contained in ∇_a mixes the index b . On the other hand $A_a A_b$ simply means the product of A_a and A_b .

problem to be solved

Suppose we have a d-dimensional curved space M and the covariant derivative $\nabla_a \ (a = 1 \sim d)$ on it.

Find

(i) a good space V and

(ii) a good object $\nabla_{(a)}$ which is equivalent to ∇_a

such that each component of $\nabla_{(a)}, (a = 1, \cdots, d)$ is expressed as a linear transformation on V.

What V we should take?

$$\nabla_a: \ \varphi(x) \mapsto \varphi_a(x),$$

$$\nabla_a: \ \varphi_a(x) \mapsto \varphi_{ab}(x), \cdots$$

$\Rightarrow$ V should be a direct sum of spaces of various fields.

We will see that the right choice is

$$V = \bigoplus_{r:\text{ irr.rep. of Spin(D)}} \underbrace{(V_r \oplus \cdots \oplus V_r)}_{d_r \ (\text{ dim. of } r\ )},$$

$$= V_{\text{regular rep.}}$$

V_r : space of field with rep. r of $spin(d)$.

$$= C^\infty(E_{prin}).$$

Answer:

Suppose $M = \bigcup_I U_I$ is a D-dim Riemannian manifold with a spin structure, whose transition functions are

$$t_{IJ}(x) \in spin(D),\ x \in U_I \cap U_J\ .$$

We assume all indices are Lorentz indices, and in particular, the covariant derivative is expressed as

$$\nabla_a = e_a^{\mu}\left(\partial_{\mu} + \omega_{\mu}^{bc} O_{bc}\right),$$

where O_{bc} is the Lorentz generator.

(i) $V = C^{\infty}(E_{prin})$

First we consider the principal $spin(D)$ bundle E_{prin} on M associated with the spin structure. In other words, we consider a direct product $U_I \times Spin(D)$ for each patch U_I, and glue them together by the following rule:

For $x \in U_I \cap U_J$,

$$(x, g_I) \in U_I \times Spin(D) \sim (x, g_J) \in U_J \times Spin(D)$$
$$\Leftrightarrow g_I = t_{IJ}(x) g_J$$

Then we take the space of functions on E_{prin} as V.

(ii) $\nabla_{(a)} = R_{(a)}{}^{b}\left(g^{-1}\right)e_b^{\mu}\left(\partial_\mu + \omega_\mu^{cd}(x)\hat{O}_{cd}\right)$

Let $R_a{}^b\left(g\right)$ be the rep. matrix of the vector rep. of $Spin(D)$, and $\hat{O}_{ab}$ be the left derivative along the fiber $Spin(D)$.

$$\varepsilon^{ab}\hat{O}_{ab}\varphi(g) = \varphi((1-\varepsilon^{ab}\tau_{ab})g) - \varphi(g).$$

Then we define a differential operator on E_{prin} by

$$\nabla_{(a)} = R_{(a)}{}^{b}\left(g^{-1}\right)e_b^{\mu}\left(\partial_\mu + \omega_\mu^{cd}(x)\hat{O}_{cd}\right).$$

We can show that each component of $\nabla_{(a)},(a=1..10)$ is a globally defined differential operator on E_{prin}, and thus a linear transformation on *V*. Therefore it can be expressed by a matrix, if we introduce UV and IR cut offs.

(Proof)

$$\begin{aligned}
\nabla_{(a)}^{[I]} &= R_{(a)}{}^{b}\left(g_I{}^{-1}\right)\nabla_b^{[I]} \\
&= R_{(a)}{}^{b}\left(g_I{}^{-1}\right)R_b{}^{c}\left(t_{IJ}\left(x\right)\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(g_I{}^{-1}t_{IJ}\left(x\right)\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(\left(t_{IJ}\left(x\right)^{-1}g_I\right)^{-1}\right)\nabla_c^{[J]} \\
&= R_{(a)}{}^{c}\left(g_J{}^{-1}\right)\nabla_c^{[J]} \\
&= \nabla_{(a)}^{[J]}
\end{aligned}$$

Furthermore we can show that each component of $\nabla_{(a)}$ is hermitian for the natural measure on E_{prin}:

$$(f,h)=\int_{E_{prin}} f^* h = \int_M d^D x\sqrt{g}\int_{spin(D)} du\, f(x,u)^* h(x,u).$$

Example 1 2D flat space $\mathbb{R}^2$

$$spin(2)=\{e^{i\theta};0\le\theta<2\pi\}\simeq S_1$$

$$E_{prin}\simeq\mathbb{R}^2\times S_1$$

$$V=C^\infty(E_{prin})=\{\varphi(x^1,x^2,\theta);R^2\times S_1\to\mathbb{C}\}$$

$$\nabla_1=\partial_1$$

$$\nabla_2=\partial_2$$

$$R_{(a)}{}^b=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$$

$$\nabla_{(1)}=\cos 2\theta\,\partial_1+\sin 2\theta\,\partial_2$$

$$\nabla_{(2)}=-\sin 2\theta\,\partial_1+\cos 2\theta\,\partial_2$$

$$\nabla_{(1)}{}^2+\nabla_{(2)}{}^2=\partial_1{}^2+\partial_2{}^2$$

Example 2 S_2 with homogeneous metric

$$spin(2) = \{e^{i\theta}; 0 \le \theta < 2\pi\} \simeq S_1$$

$$E_{prin} \simeq S_1 \ bundle \ over \ S_2 \simeq S_3 \quad \text{(Hopf bundle)}$$

$$V = C^{\infty}(E_{prin}) = \{\varphi(z,\theta); S_3 \to \mathbb{C}\}$$

z: the stereographic coordinate of S_2

$$\nabla_{(+)} = e^{-2i\theta}\{(1+z\bar{z})\partial_z + \frac{i}{2}\bar{z}\partial_\theta\}, \qquad \leftarrow \qquad \nabla_{+} = (1+z\bar{z})\partial_z + \frac{i}{2}\bar{z}O_{12}$$

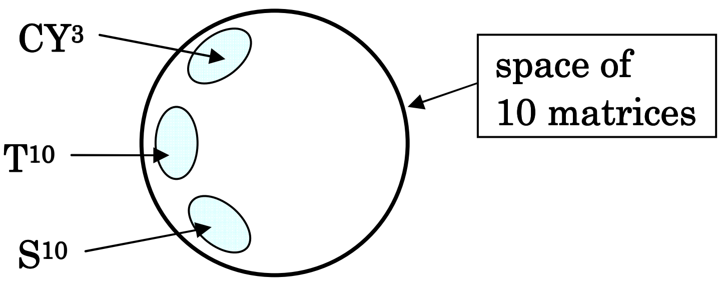
$$\nabla_{(-)} = e^{2i\theta}\{(1+z\bar{z})\partial_{\bar{z}} - \frac{i}{2}z\partial_\theta\}, \qquad \nabla_{-} = (1+z\bar{z})\partial_{\bar{z}} - \frac{i}{2}zO_{12}$$

where $\pm = 1 \pm i2$.

Each of $\nabla_{(+)}$ and $\nabla_{(-)}$ is a globally defined differential operator on E_{prin}.

In this manner, S_2 is realized in terms of two matrices. This should be distinguished from the ordinary fuzzy sphere, which is obtained by embedding to the space of three matrices.

Similarly, any Riemannian manifold with dimension less than or equal to D can be coded in the space of D matrices.



V is the space of regular representation field.

$$V = \bigoplus_{r:\text{ rep. of Spin(D)}} \underbrace{(V_r \oplus \cdots \oplus V_r)}_{d_r\ (\text{ dim. of } r\ )}, \quad V_r : \text{space of a field with rep. } r.$$

scalar, spinor, vector,…etc.

(∵) Since an element of V is a function on E_{prin}, at each point on M, it gives a function $Spin(D) \to \mathbb{C}$.

In general, the space of functions on a group G forms a special representation called the regular representation, which is isomorphic to

$$v_{reg} \cong \bigoplus_{r:\text{irr.rep. of G}} \underbrace{(v_r \oplus \cdots \oplus v_r)}_{d_r},$$

where v_r is a representation of G, and d_r is its dimension.

$C^\infty(G)$ can be spaned by $R_{<r>}{}^{i,j}(g)$,
where $R_{<r>}{}^{i,j}(g)$ is the rep. matrix for r.

The action of an element $h \in G$ on $f \in C^\infty(G)$ is given by

$$f(g) \mapsto f(h^{-1}g).$$

In particular,

$$R_{<r>}{}^{i,j}(g) \mapsto R_{<r>}{}^{i,j}(h^{-1}g) = \sum_k R_{<r>}{}^{i,k}(h^{-1}) R_{<r>}{}^{k,j}(g).$$

The regular rep. has the following remarkable property:

$$v_{reg} \otimes v_r \cong v_{reg} \oplus \cdots \oplus v_{reg} \ \text{, for any } r.$$

More explicitly, this isomorphism is given by

$$f^i(g) \in v_{reg} \otimes v_r \mapsto f^{(i)}(g) = R_{<r>}{}^{ij}(g^{-1}) f^j(g).$$

Indeed $f^{(i)}(g)$ transforms under $h \in G$ as

$$f^{(i)}(g) \mapsto$$
$$R_{<r>}{}^{ij}(g^{-1}) R_{<r>}{}^{jk}(h) f^k(h^{-1}g) = R_{<r>}{}^{ik}(g^{-1}h) f^k(h^{-1}g) = f^{(i)}(h^{-1}g).$$

Therefore we have

$$\nabla_a : V \to V \otimes T \cong V \oplus \cdots \oplus V,$$

where T is the tangent bundle, and the combined map is given by

$$\nabla_{(a)} = R_{(a)}{}^b \left(g^{-1}\right) e_b^\mu \left(\partial_\mu + \omega_\mu^{cd}(x) \hat{O}_{cd}\right).$$

New interpretation of IIB matrix model

We now regard the matrices in IIB matrix model as linear transformations on $C^{\infty}(E_{prin})$.

Here we consider the classical EOM derived from the action

$$S = -\frac{1}{4} Tr\left(\left[A_a, A_b\right]^2\right) + fermions .$$

If we set the fermions to be zero, it becomes

$$\left[A_a\left[A_a, A_b\right]\right] = 0 .$$

Now we can impose the following Ansatz

$$A_a = i\nabla_{(a)} ,$$

because $\nabla_{(a)}$ is a well defined linear transformation, and we have

$$\left[\nabla_{(a)}\left[\nabla_{(a)}, \nabla_{(b)}\right]\right] = 0 .$$

Let's rewrite this equation in terms of the ordinary covariant derivative ∇_a.

Formula

$$\begin{aligned}\nabla_{(a)}\nabla_{(b)} &= R_{(a)}{}^{c}\left(g^{-1}\right)\nabla_c\, R_{(b)}{}^{d}\left(g^{-1}\right)\nabla_d \\ &= R_{(a)}{}^{c}\left(g^{-1}\right) R_{(b)}{}^{d}\left(g^{-1}\right)\nabla_c \nabla_d\end{aligned}$$

In the last expression, the Lorentz generator in ∇_c acts on the index d of ∇_d.

Using this, we have

$$0=\left[\nabla_{(a)},\left[\nabla_{(a)},\nabla_{(b)}\right]\right]$$

$$\Leftrightarrow$$

$$0=\left[\nabla_a,\left[\nabla_a,\nabla_b\right]\right]$$

$$=\left[\nabla_a,R_{ab}{}^{cd}O_{cd}\right]=(\nabla_a R_{ab}{}^{cd})O_{cd}-R_{ab}{}^{ca}\nabla_c$$

$$\Leftrightarrow \nabla_a R_{ab}{}^{cd}=0\,,\ R_{ab}=0$$

$$\Leftrightarrow R_{ab}=0\,.$$

The Einstein equation follows from the EOM of IIB matrix model.

If we start with IIB action with a mass term

$$S'=-\frac{1}{4}Tr\left(\left[A_a,A_b\right]^2\right)+\frac{m^2}{2}Tr\left(A_a{}^2\right)+fermions\ ,$$

we have the Einstein equation with a cosmological constant

$$R_{ab}=-m^2\delta_{ab}\quad .$$

Diffeomorphism and local Lorentz invariance

We now show that the symmetries of the general relativity are realized as parts of the SU(N) symmetry of the matrix model.

In the matrix model the infinitesimal SU(N) symmetry is given by

$$\delta A_a = [\Lambda, A_a].$$

If we interpret V as $C^\infty(E_{prin})$, we can take various elements of $End(C^\infty(E_{prin}))$ as Λ.

(1) diffeomorphism

$$\Lambda = \frac{1}{2}\{\lambda^{(a)}(x,g), \nabla_{(a)}\} \in End(C^\infty(E_{prin})) \sim \lambda^a(x)\nabla_a$$

$$\lambda_{(a)}(x,g) = R_{(a)}{}^b\left(g^{-1}\right)\lambda_b(x)$$

$\Rightarrow \delta A_a = [\Lambda, A_a]$ correctly reproduces the diffeomorphism.

(2) local Lorentz

$$\Lambda = \lambda^{ab}(x)\hat{O}_{ab} \in End(C^\infty(E_{prin}))$$

$\Rightarrow \delta A_a = [\Lambda, A_a]$ is the local Lorentz transformation.

fluctuations around the configuration

fluctuations around $A_\mu^{(0)} = i\partial_\mu$	$\Rightarrow$ quenching	Large-N Yang-Mills theory
fluctuations around $A_\mu^{(0)} = i\nabla_{(a)}$	$\Rightarrow$ quenching	Large-N Yang-Mills theory on M?

For simplicity, let's consider a scalar matrix:

$$S = N\,Tr\left(-\frac{1}{2}\left[A_\mu^{(0)}, \phi \right]^2 + \frac{\lambda}{3}\phi^3 \right) .$$

For $A_\mu^{(0)} = i\partial_\mu$, we have in the coordinate representation

$$S = N\frac{1}{2}\int d^d x\, d^d y \left(\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) \phi(x, y) \right)^2$$

$$+ N\frac{\lambda}{3}\int d^d x\, d^d y\, d^d z\, \phi(x, y)\phi(y, z)\phi(z, x),$$

where $\phi(x, y) = \langle x | \phi | y \rangle$.

This looks like a non-local field, but it turns out to be equivalent to the large-N matrix model.

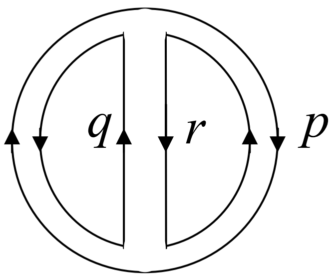
In the momentum representation, we have

$$S = N\frac{1}{2}\int d^d p\, d^d q \left(p_\mu - q_\mu\right)^2 \left|\tilde{\phi}(p,q)\right|^2$$

$$+ N\frac{\lambda}{3}\int d^d p\, d^d q\, d^d r\, \tilde{\phi}(p,q)\tilde{\phi}(q,r)\tilde{\phi}(r,p),$$

where $\tilde{\phi}(p,q) = \langle p|\phi|q\rangle$.

Then it is easy to check that the Feynman diagrams are the same as those of the large-N matrix model.



For $A_\mu^{(0)} = i\nabla_{(a)}$, if the space-time is flat, the g dependence is factored out as $n \times n$ matrix:

$$S = N\frac{1}{2}\int d^d x\, d^d y \int dg dg'\, \phi(x,g,y,g')\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu}\right)^2 \phi(x,g',y,g)$$

$$+ N\frac{\lambda}{3}\int d^d x\, d^d y\, d^d z \int dg dg' dg''\phi(x,g,y,g')\phi(y,g',z,g'')\phi(z,g'',x,g),$$

$$= N\frac{1}{2}\int d^d x\, d^d y\ \mathrm{tr}\left(\phi(x,y)\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu}\right)^2 \phi(x,y)\right)$$

$$+ N\frac{\lambda}{3}\int d^d x\, d^d y\, d^d z\ \mathrm{tr}\left(\phi(x,y)\phi(y,z)\phi(z,x)\right).$$

Here we have assumed that $C^\infty(spin(d))$ is cutoff to n dimensional space.

This is equivalent to the flat case with internal degrees of freedom $A_\mu^{(0)} = i\partial_\mu \otimes 1_n$, which is again equivalent to the large-N matrix model.

For flat space-time, the new interpretation $A_\mu^{(0)} = i\nabla_{(a)}$ is not different from the old one $A_\mu^{(0)} = i\partial_\mu$, provided that $C^\infty(spin(d))$ is cutoff to a smaller dimensions compared to N.

It is interesting to see how the curvature of space-time couples to the Yang-Mills theory.

Summary

- Any d-dimensional manifold can be embedded in d matrices.
- The matrices in IIB matrix model can be interpreted as differential operators on the principal bundle on any manifold of less than or equal to 10 dimensions.
- In this interpretation, any manifold of less than or equal to 10 dimensions is included in the path integral of IIB matrix model.
- The symmetries of the general relativity are realized as parts of the SU(N) symmetry of the matrix model.
- The classical EOM of IIB matrix model gives the Einstein equation.
- It would be interesting to see what theory emerges if the fluctuations are taken into account.

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- In order to implement SUSY, we can consider super manifold instead of the ordinary manifold, and consider the matrices as differential operators on the associated principal bundle. Then V becomes a super vector space and the matrices should be regarded as super matrices, but the form of the action of IIB matrix model seems to work without any modification.