Phases of $\mathcal{N}=2$ theories in 1+1 dimensions with boundaries

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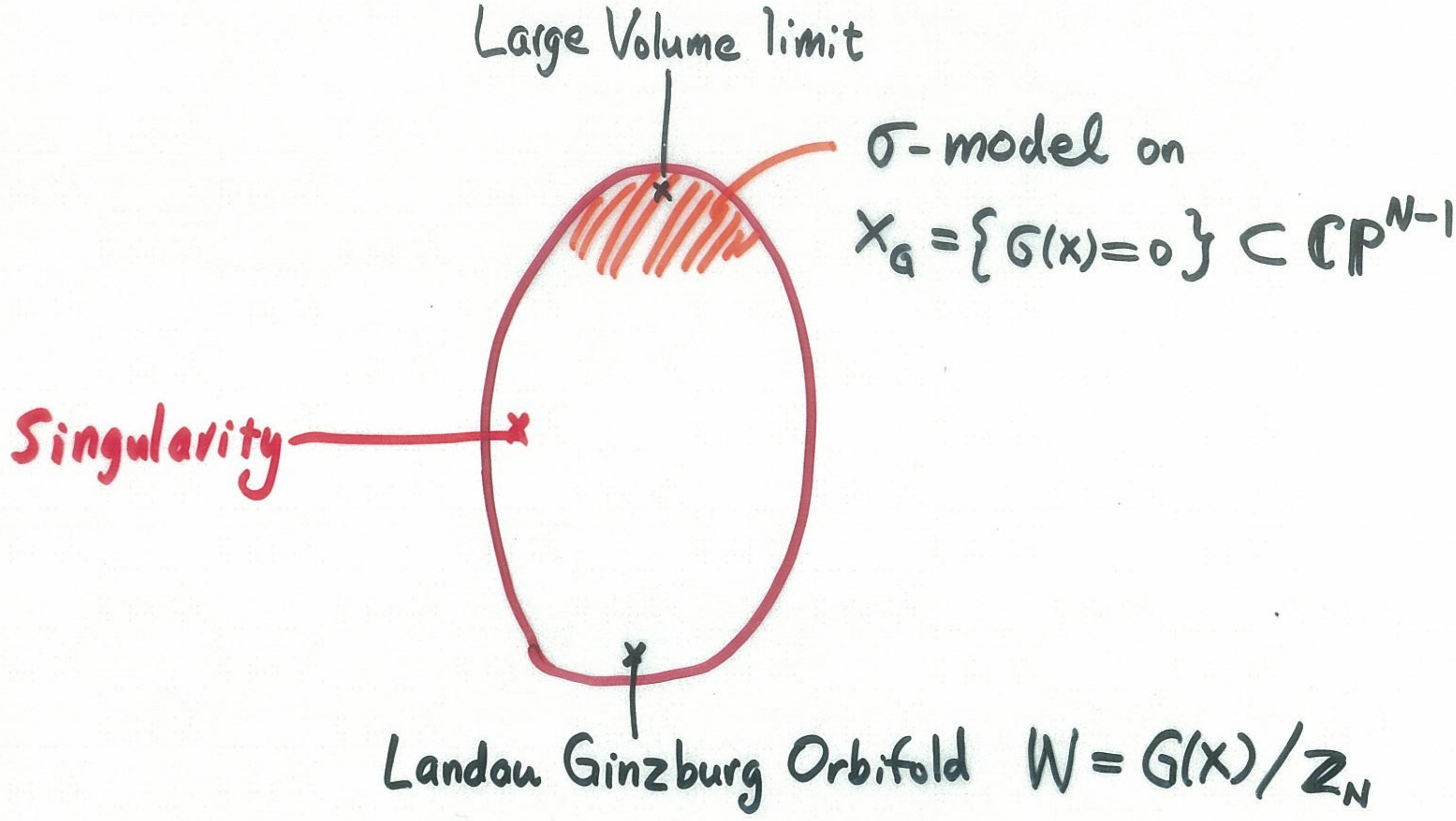
$G(X) = G(x_1, \dots, x_N)$ degree N polynomial (e.g. $x_1^N + \dots + x_N^N$)

$\rightsquigarrow$ a family of 2d (2,2) SUSY QFTs

parametrized by $\underline{\mathcal{M}_K}$



This picture is obtained by

- Mirror Symmetry Candelas, de la Ossa, Green, Parkes
- Linear Sigma Model Witten

$\mathcal{N}=2_B$ invariant D-branes in these QFTs:

complex of holomorphic vector bundles over X_G

$$\cdots \; \mathcal{E}_{-1} \xrightarrow{T_{-1}} \mathcal{E}_0 \xrightarrow{T_0} \mathcal{E}_1 \xrightarrow{T_1} \cdots \qquad T_{i+1}\circ T_i = 0$$

$\mathbb{Z}$ -invariant & homogeneous

matrix factorizations of $W = G(X)$

$$f(x)\,g(x) = g(x)\,f(x) = G(x)\cdot\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$r \times r$ matrices (r: any)

AIM: Construct a family of boundary QFTs that relates D-branes at different points of $\mathcal{M}_K$

Bulk LSM

E. Witten 1979, 1993

(2,2) SUSY gauge theory in 1+1 dimensions

Gauge group $U(1)$, matter field $P, X_1, \cdots, X_N$ chiral

charge $-N, 1, \cdots, 1$

superpotential $W = PG(X_1, \cdots, X_N)$

parameters

$$t = r - i\theta$$

r: F.I. $\quad$ θ: Theta $\qquad \mathcal{M}_K$

$$G(x) = \sum a_{i_1 \cdots i_N} X_{i_1} \cdots X_{i_N} \qquad \mathcal{M}_C$$

$\mathbb{C}$-scalars $\quad p, x_1, \cdots, x_N$ from chirals

σ from vector $\quad \bar{D}_+ D_- V = \sigma + \theta^+ \lambda_+ + \bar{\theta}^- \lambda_- + \theta^+ \bar{\theta}^- (D - iV_{\cdots}$

potential

$$U = |-N\sigma p|^2 + \|\sigma x\|^2$$

$$+ \frac{e^2}{2}\left(-N|p|^2 + \|x\|^2 - r\right)^2 \quad \leftarrow D$$

$$+ |p|^2 \left\|\frac{\partial G}{\partial x}\right\|^2 + |G(x)|^2 \quad \leftarrow F$$

$r = -\infty \longleftarrow \; r = 0 \; \longrightarrow r = +\infty$

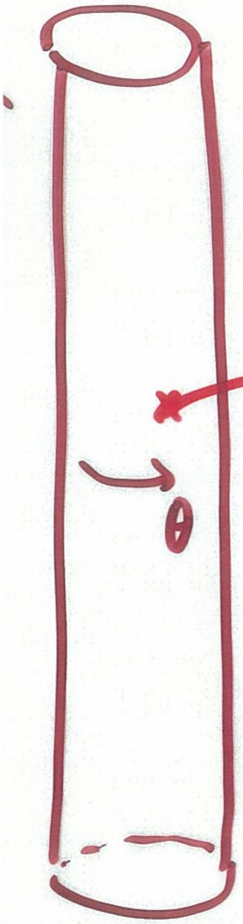
$|p|^2 \neq 0 : U(1) \to \mathbb{Z}_N \qquad \qquad \|x\|^2 \neq 0 : U(1) \to \{1\}$

Low energy theory

$\boxed{r \gg 0}$ $\quad U=0 \iff \sigma=p=0,\ \|x\|^2=r,\ G(x)=0$

all transverse modes heavy

$\Rightarrow$ **σ-model on** $\underbrace{\left\{ x\in\mathbb{C}^N \,\middle|\, \begin{matrix}\|x\|^2=r\\ G(x)=0\end{matrix} \right\} / U(1)}_{\substack{\parallel \\ X_G}}$



$$\boxed{\begin{aligned} r_{eff} &= r - N\log N = 0 \\ \theta_{eff} &= \theta + N\pi \equiv 0 \quad \mathrm{mod}\ 2\pi\mathbb{Z} \end{aligned}}$$

$$U_{eff}(\sigma) \overset{|\sigma|\ \text{large}}{=\!=} \frac{e^2}{2} r_{eff}^2 + \overbrace{\frac{e^2}{2}(\theta_{eff}+2\pi n)^2}^{\text{electrostatic energy}} = 0$$

flat direction in σ

Singularity !

$\boxed{r \ll 0}$ $\quad U=0 \iff p=\sqrt{\frac{-r}{N}},\ x=0$

Only $X_1, \cdots, X_N$ are massless

$\Rightarrow$ **LG orbifold** $W=\langle p\rangle G(X)/\mathbb{Z}_N$

D-branes in LSM (classical theory)

Data: Gauge invariant & homogeneous matrix factorizations of $W = pG(x)$

$$Q(p,x) = \begin{pmatrix} 0 & f(p,x) \\ g(p,x) & 0 \end{pmatrix}_{2r\times 2r}, \quad \underline{Q^2 = W\mathbf{1}_{2r}}$$

$$\rho(z) = \begin{pmatrix} z^{q_1} & & \\ & \ddots & \\ & & z^{q_{2r}} \end{pmatrix} \quad q_i \in \mathbb{Z}$$

gauge group representation of Chan-Paton space

$$R(\lambda) = \begin{pmatrix} \lambda^{R_1} & & \\ & \ddots & \\ & & \lambda^{R_{2r}} \end{pmatrix}, \quad R(-1) = \begin{pmatrix} \mathbf{1}_r & \\ & -\mathbf{1}_r \end{pmatrix}$$

R-symmetry action on CP space

s.t. $$\underline{\rho(z)^{-1} Q(z^{-N}p, zx)\rho(z) = Q(p,x)}$$

$$\underline{R(\lambda)\, Q(\lambda^2 p, x)\, R(\lambda)^{-1} = \lambda Q(p,x)}$$

$\Longrightarrow$ **Brane-Antibrane** system with tachyon $Q + Q^\dagger$

$$\bigoplus_{i=1}^{r} \mathcal{W}(q_i) \underset{f+g^\dagger}{\overset{g+f^\dagger}{\rightleftarrows}} \bigoplus_{i=r+1}^{2r} \mathcal{W}(q_j)$$

$\mathcal{W}(q)$ .... Wilson line brane $\underline{\rho(z) = z^q}$

i.e. boundary interaction $P\exp\left(-i\int_{\partial\Sigma} \mathcal{A}_t\, dt\right)$

$$\boxed{\mathcal{A}_t = \rho_*(v_0 - \mathrm{Re}(\sigma)) + \frac{1}{2}\{Q, Q^\dagger\} - \frac{1}{2}\sum(\psi^i \partial_i Q + \text{c.c.})}$$

sum over $P, X_1, \cdots, X_N$

Gauge transformation

$$i\mathcal{A}_t \longrightarrow \rho(z)\, i\mathcal{A}_t\, \rho(z)^{-1} + \rho(z)\, \partial_t \rho(z)^{-1}$$

R-symmetry

$$i\mathcal{A}_t \longrightarrow R(\lambda)\, i\mathcal{A}_t\, R(\lambda)^{-1}$$

$\mathcal{N}=2_B$ supersymmetry

$$\delta\mathcal{A}_t = -\mathrm{Re}\Big\{\sum \bar\epsilon \psi^i \partial_i Q^2 - [\epsilon Q^\dagger, Q^2]\Big\} + i\mathcal{D}_t(\bar\epsilon Q + \epsilon Q^\dagger) - i(\dot{\bar\epsilon} Q + \dot\epsilon Q^\dagger)$$

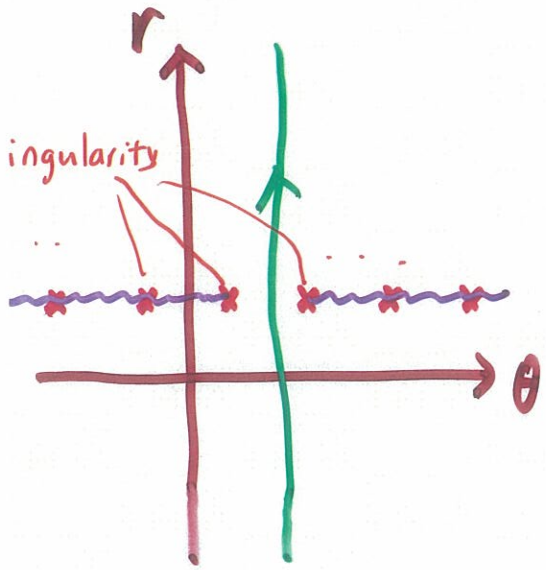
0 if $Q^2 \propto \mathbb{1}$

if $Q^2 = W\mathbb{1}$ cancels the "Warner term"

$$\delta S_{\text{bulk}} = -\mathrm{Re}\int_{\partial\Sigma} dt \sum \bar\epsilon \psi^i \partial_i W$$

The Grade Restriction Rule "GRR"

B-branes : θ is not periodic $\left[\int_\Sigma \frac{i}{2\pi} F_A \notin \mathbb{Z} \right]$



Draw cuts with a window of width 2π

Consider paths that go through that window

GRR : Along such a path, we only admit LSM branes based on $\mathcal{W}(q)$'s with

$$\boxed{-\frac{N}{2} < \frac{\theta}{2\pi} + q < \frac{N}{2}}$$

for any θ in that window.

i.e. gauge charges of the CP states must be in this bound.

e.g. $N=5$ window $-\pi < \theta < \pi \Rightarrow q \in \{0, \pm 1, \pm 2\}$

window $-3\pi < \theta < -\pi \Rightarrow q \in \{-1, 0, 1, 2, 3\}$

"category" to be precise

I_{window} := the "set" of grade restricted LSM branes ⋯ depends on window

Then, we have :

LSM — all LSM branes

$$\begin{tikzcd}[column sep=small]
 & MF_{\mathbb{C}^\times}(pG(x)) \arrow[d, phantom, "\cup" description] & \\
 & I_{\text{window}} \arrow[dl, "\approx"'] \arrow[dr, "\approx"] & \\
MF_{\mathbb{Z}_N}(G(x)) & & D^b(\mathrm{Coh}\, X_G)
\end{tikzcd}$$

grade restricted ones

LG orbifold: $MF_{\mathbb{Z}_N}(G(x))$

CY σ-model: $D^b(\mathrm{Coh}\, X_G)$

$MF_{\mathbb{Z}_N}(G(x))$ ↑ $\mathbb{Z}_N$-invariant, homogeneous matrix factorizations of $G(x)$

$D^b(\mathrm{Coh}\, X_G)$ ↑ Complexes of vector bundles over $X_G \subset \mathbb{CP}^{N-1}$

LSM → LG "set $p=1$"

(V, Q, ρ, R) a LSM brane $\left[\ Q(p,x)^2 = pG(x)\,\mathrm{id}_V, \cdots, \cdots\ \right]$

Set $\bar{Q}(x) = Q(1,x)$, $\bar{Q}(x)^2 = G(x)\,\mathrm{id}_V$ ✓ m.f. of $G(x)$

$$\omega^N = 1 \Rightarrow \rho(\omega)^{-1}\bar{Q}(\omega x)\rho(\omega) = \rho(\omega)^{-1}Q(\underset{\omega^{-N}1}{\underset{\shortparallel}{1}}, \omega x)\rho(\omega) = Q(1,x) = \bar{Q}(x)$$

✓ $\mathbb{Z}_N$-invariance

(Set $\bar{\rho}(\omega) = \rho(\omega)$.)

$$R(\lambda)\underbrace{Q(\lambda^2 1, x)}_{\shortparallel}R(\lambda)^{-1} = \lambda Q(1,x)$$

$$\rho(\lambda^{2/N})^{-1}Q(1, \lambda^{2/N}x)\rho(\lambda^{2/N}) \qquad \boxed{\therefore\ \bar{R}(\lambda) = R(\lambda)\rho(\lambda^{2/N})^{-1}} \Rightarrow$$

$$\bar{R}(\lambda)\bar{Q}(\lambda^{2/N}x)\bar{R}(\lambda)^{-1} = \lambda\bar{Q}(x)$$ ✓ R-symmetry

Note $\bar{R}(e^{\pi i})\bar{\rho}(e^{2\pi i/N}) = R(-1) = \sigma_V$ ···· required integrality in LG orbifold

Thus $(V, \bar{Q}, \bar{\rho}, \bar{R})$ is a brane of the LG orbifold $W = G(x)/\mathbb{Z}_N$

LG → LSM with GRR

[Note : $S = N$]

$(V, \bar{Q}, \bar{\rho}, \bar{R})$ a brane of LGO

$$\left[\begin{array}{l} \bar{R}(\lambda) = \begin{pmatrix} \lambda^{\bar{R}_1} & & \\ & \ddots & \\ & & \lambda^{\bar{R}_{2r}} \end{pmatrix} \\ \bar{R}(e^{\pi i})\bar{\rho}(e^{2\pi i/N}) = \begin{pmatrix} \mathbb{1}_r & \\ & -\mathbb{1}_r \end{pmatrix} \end{array}\right]$$

$$\left.\begin{array}{lll} \exists.1 \ R_i \in 2\mathbb{Z} & i = 1 \cdots r \\ \qquad\quad 2\mathbb{Z}+1 & i = r+1, \cdots, 2r \\ \exists.1 \ q_i \in \mathbf{N}_{\text{window}} & i = 1 \cdots 2r \end{array}\right\} \text{s.t.} \quad \boxed{\bar{R}_i = R_i - \frac{2q_i}{N}}$$

define $\boldsymbol{\rho}(g) := \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_{2r}} \end{pmatrix}$, $\mathbf{R}(\lambda) := \begin{pmatrix} \lambda^{R_1} & & \\ & \ddots & \\ & & \lambda^{R_{2r}} \end{pmatrix}$

Then $\bar{R}(\lambda) = R(\lambda)\,\rho(\lambda^{2/N})^{-1}$

In particular, $\bar{\rho}(e^{2\pi i/N}) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \bar{R}(e^{\pi i})^{-1} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R(e^{\pi i})}_{\shortparallel \ 1} \rho(e^{2\pi i/N})$

Orbifold invariance $\Rightarrow$ $\rho(e^{2\pi i/N})^{-1}\,\bar{Q}(e^{2\pi i/N}x)\,\rho(e^{2\pi i/N}) = \bar{Q}(x)$

$\therefore$ $\rho(z)^{-1}\bar{Q}(zx)\rho(z)$ is a function of z^N ($\&$ x).

$\bar{Q}(zx)$ … polynomial in z

$\rho(z)^{-1}(\cdots)\rho(z)$ can change powers of z at most by $z^{\pm(N-1)}$

$\therefore$ polynomial in z^N (no negative power)

$$\rho(z)^{-1}\bar{Q}(zx)\rho(z) = \bar{Q}_0(x) + z^N\bar{Q}_1(x) + z^{2N}\bar{Q}_2(x) + \cdots$$

Now put

$$\mathbf{Q}(p,x) := \bar{Q}_0(x) + p\bar{Q}_1(x) + p^2\bar{Q}_2(x) + \cdots$$

Then $\rho(g)^{-1}Q(g^{-N}p, gx)\rho(g) = \bar{Q}_0(x) + (g^{-N}p)\cdot g^N\bar{Q}_1(x) + \cdots$

$$= Q(p,x)$$

✓ Gauge invariance

$$R(\lambda)Q(\lambda^2 p, x)R(\lambda)^{-1} = \bar{R}(\lambda)\underbrace{\rho(\lambda^{2/N})Q(\lambda^2 p, x)\rho(\lambda^{2/N})^{-1}}_{\substack{\| \\ Q(p,\lambda^{2/N}x)}}\bar{R}(\lambda)^{-1}$$

$$= \lambda Q(p,x)$$

✓ R-symmetry

$$\bar{Q}(x)^2 = G(x)\,\mathrm{id}_V \rightsquigarrow Q(p,x)^2 = pG(x)\,\mathrm{id}_V$$

✓ mat. fac. of $pG(X)$

Thus, we obtain a LSM brane (V, Q, ρ, R)

grade restricted

LSM $\to$ σ-model

At $r \gg 0$, P & transverse to $G(X)=0$ are heavy

$\to$ integrate them out!

But p is in $Q(p, X)$ .... boundary interaction.

Only bulk modes of P are integrated out.

$\rightsquigarrow$ effective theory including $P|_{\partial\Sigma}$.

We find that we have a 1st order system

$$\boxed{L_{eff} = \int_{\partial\Sigma} i p \overleftrightarrow{D}_t \bar{p} \, dt + \cdots}$$

$\Rightarrow$ $[\bar{p}, p] = 1$ $\qquad$ p creation, $\bar{p}$ annihilation

represented on ∞-dim Fock space

	$\lvert 0\rangle$,	$p\lvert 0\rangle$,	$p^2\lvert 0\rangle$,	$\cdots$	$p^k\lvert 0\rangle$,	$\cdots$
gauge charge	0	N	$2N$	$\cdots$	kN	$\cdots$
R-charge	0	2	4	$\cdots$	$2k$	$\cdots$

Total CP space $\mathbb{C}^{2r} \otimes \bigoplus_{k=0}^{\infty} P^k |0\rangle = \bigoplus_{i=1}^{2r} \bigoplus_{k=0}^{\infty} \underbrace{\mathbb{C}_{(i)} \otimes P^k |0\rangle}$

gauge charge $q_i + Nk$

line b'dle $\mathcal{O}(q_i + Nk)$

R-charge $R_i + 2k$

Collect those with R-charge $= j$:

$$\mathcal{E}^j := \bigoplus_{\substack{(i,k) \\ j = R_i + 2k}} \mathcal{O}(q_i + Nk)$$

$j_* := \min_i \{R_i\}$

$$R(\mu)\, Q(\mu^2 p, x) = \mu\, Q(p, x)\, R(\mu) \quad \Rightarrow$$

$$\boxed{\cdots 0 \to 0 \to \mathcal{E}^{j_*} \xrightarrow{Q} \mathcal{E}^{j_*+1} \to \cdots \xrightarrow{Q} \mathcal{E}^{j} \xrightarrow{Q} \mathcal{E}^{j+1} \xrightarrow{Q} \cdots}$$ (☆)

$Q^2 : \mathcal{E}^j \xrightarrow{pG(x)\times} \mathcal{E}^{j+2}$ is zero over X_G.

$\therefore$ (☆) is a complex of vector bundles.

- ∞-length toward right
- but $\mathcal{E}^j \xrightarrow{Q} \mathcal{E}^{j+1} \xrightarrow{Q} \mathcal{E}^{j+2}$ exact at $j \gg 0$.

(☆) $\overset{\text{qis}}{\approx}$ finite length complex.

Example $\quad N=5 \quad G(X)=X_1^5+X_2^5+X_3^5+X_4^5+X_5^5$

At LG : $\bar{Q}=\sum_{i=1}^{5}\left(\underset{\frac{2}{5}}{X_i}\,\underset{\frac{3}{5}}{\eta_i}+X_i^4\,\underset{-\frac{3}{5}}{\bar{\eta}_i}\right)$ $\quad \{\eta_i,\bar{\eta}_j\}=\delta_{ij}$

$\{\eta_i,\eta_j\}=\{\bar{\eta}_i,\bar{\eta}_j\}=0$

represented on $\underbrace{\overset{(1)}{|0\rangle}\ \overset{(10)}{\bar{\eta}_i\bar{\eta}_j|0\rangle},\ \overset{(5)}{\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k\bar{\eta}_\ell|0\rangle}}_{\text{even}},\ \underbrace{\overset{(5)}{\bar{\eta}_i|0\rangle},\ \overset{(10)}{\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k|0\rangle},\ \overset{(1)}{\bar{\eta}_1\cdots\bar{\eta}_5|0\rangle}}_{\text{odd}}$

R-charge : $\tilde{R}_0 \quad \tilde{R}_0-\frac{6}{5} \quad \tilde{R}_0-\frac{12}{5} \qquad \tilde{R}_0-\frac{3}{5} \quad \tilde{R}_0-\frac{9}{5} \quad \tilde{R}_0-\frac{15}{5}$

.... **L=0 RS branes** $(M=\tilde{R}_0\cdot 5)$

To Large volume : e.g. $\mathcal{N}_{\text{Window}}=\{0,1,2,3,4\}$

$\boxed{\tilde{R}_0=0}$ $\quad$ first solve $\tilde{R}_i=R_i-\frac{2q_i}{5}$ $\quad \left(\begin{array}{l} R_i\in 2\mathbb{Z}/2\mathbb{Z}+1 \\ q_i\in N_\theta \end{array}\right.$

$\lvert 0\rangle$	$0=0-\frac{2\cdot 0}{5}$	$\Rightarrow R=0$	$q=0$	$\mathcal{O}(0)$	at $j=0$
$\bar{\eta}_i\bar{\eta}_j\lvert 0\rangle$	$-\frac{6}{5}=0-\frac{2\cdot 3}{5}$	$\Rightarrow R=0$	$q=3$	$\mathcal{O}(3)^{\oplus 10}$	at $j=0$
$\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k\bar{\eta}_\ell\lvert 0\rangle$	$-\frac{12}{5}=-2-\frac{2\cdot 1}{5}$	$\Rightarrow R=-2$	$q=1$	$\mathcal{O}(1)^{\oplus 5}$	at $j=-2$
$\bar{\eta}_i\lvert 0\rangle$	$-\frac{3}{5}=1-\frac{2\cdot 4}{5}$	$\Rightarrow R=1$	$q=4$	$\mathcal{O}(4)^{\oplus 5}$	at $j=1$
$\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k\lvert 0\rangle$	$-\frac{9}{5}=-1-\frac{2\cdot 2}{5}$	$\Rightarrow R=-1$	$q=2$	$\mathcal{O}(2)^{\oplus 10}$	at $j=-1$
$\bar{\eta}_1\cdots\bar{\eta}_5\lvert 0\rangle$	$-\frac{15}{5}=-3-\frac{2\cdot 0}{5}$	$\Rightarrow R=-3$	$q=0$	$\mathcal{O}(0)$	at $j=\overset{j_{min}}{-3}$

$$
\begin{tikzcd}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \\
 & & & & \mathcal{O}(0) \arrow[r, "x^4"] \arrow[dr, "x"] & \mathcal{O}(4)^5 \arrow[r, "x"] \arrow[dr, "x^4"] & \mathcal{O}(5) \arrow[r] & {} \\
 & & & \mathcal{O}(2)^{10} \arrow[r, "x"] \arrow[dr, "x^4"] & \mathcal{O}(3)^{10} \arrow[r, "x^4"] & \mathcal{O}(7) \arrow[r, "x"] \arrow[dr, "x^4"] & \mathcal{O}(8)^{10} \arrow[r] & \cdots \\
0 \arrow[r] \arrow[d] & \mathcal{O}(0) \arrow[r, "x"] \arrow[d] & \mathcal{O}(1)^5 \arrow[r, "x^4"] \arrow[ur, "x"] \arrow[d] & \mathcal{O}(5) \arrow[r, "x"] \arrow[d] & \mathcal{O}(6)^5 \arrow[r, "x^4"] \arrow[ur, "x"] \arrow[d, "pr"] & \mathcal{O}(10) \arrow[r, "x"] \arrow[d] & \mathcal{O}(11)^5 \arrow[r] \arrow[d] & {} \wr\, qis \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \boxed{\mathcal{O}(0)} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

$$\boxed{\widetilde{R}_0 = -\frac{2}{5}}$$

$$
\begin{tikzcd}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \\
 & & & \mathcal{O}(0)^5 \arrow[r, "x"] \arrow[dr] & \mathcal{O}(1) \arrow[r] & \mathcal{O}(5)^5 \arrow[r] \arrow[dr] & \mathcal{O}(6) \arrow[r] & \cdot \\
 & & & \mathcal{O}(3)^{10} \arrow[r] \arrow[dr] & \mathcal{O}(4)^{10} \arrow[r] \arrow[ur] & \mathcal{O}(8)^{10} \arrow[r] \arrow[dr] & \mathcal{O}(9)^{10} \arrow[r] & {} \\
0 \arrow[r] & \mathcal{O}(1) \arrow[r, "x"] & \mathcal{O}(2)^5 \arrow[r, "x^4"] \arrow[ur, "x"] & \mathcal{O}(6) \arrow[r, "x"] \arrow[d] & \mathcal{O}(7)^5 \arrow[r, "x^4"] \arrow[ur] \arrow[d] & \mathcal{O}(11) \arrow[r, "x"] & \mathcal{O}(12)^5 \arrow[r] \arrow[ur] & \cdots \wr\, qis \\
\cdots & 0 \arrow[r] & 0 \arrow[r] & \mathcal{O}(0)^5 \arrow[r, "x"] & \mathcal{O}(1) \arrow[r] & 0 & \cdots & \\
\cdots \arrow[r] & 0 \arrow[r] & \boxed{T^*_{p^*}(1)} \arrow[r] \arrow[ur] & 0 \arrow[r] & 0 & \cdot & &
\end{tikzcd}
$$

$\boxed{\widetilde{R}_0 = -\frac{4}{5}}$ $\quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad \cdots$

$$\begin{array}{l}
\xrightarrow{x} \mathcal{O}(1)^5 \xrightarrow{x} \mathcal{O}(2) \to \mathcal{O}(6)^5 \to \\
\mathcal{O}(0)^{10} \to \mathcal{O}(4)^{10} \to \mathcal{O}(5)^{10} \to \mathcal{O}(9)^{10} \to \quad \cdots \\
0 \to \mathcal{O}(2) \xrightarrow{x} \mathcal{O}(3)^5 \xrightarrow{x^4} \mathcal{O}(7) \to \mathcal{O}(8)^5 \to \mathcal{O}(12) \to \\
\quad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \wr\wr \text{ q.is} \\
0 \to \mathcal{O}(0)^{10} \xrightarrow{x} \mathcal{O}(1)^5 \xrightarrow{x} \mathcal{O}(2) \to 0 \\
\qquad \wr\wr \text{ q.is} \\
\cdots \to 0 \to \boxed{\Lambda^2 T^*_{\mathbb{P}^4}(1)} \to 0 \to \cdots
\end{array}$$

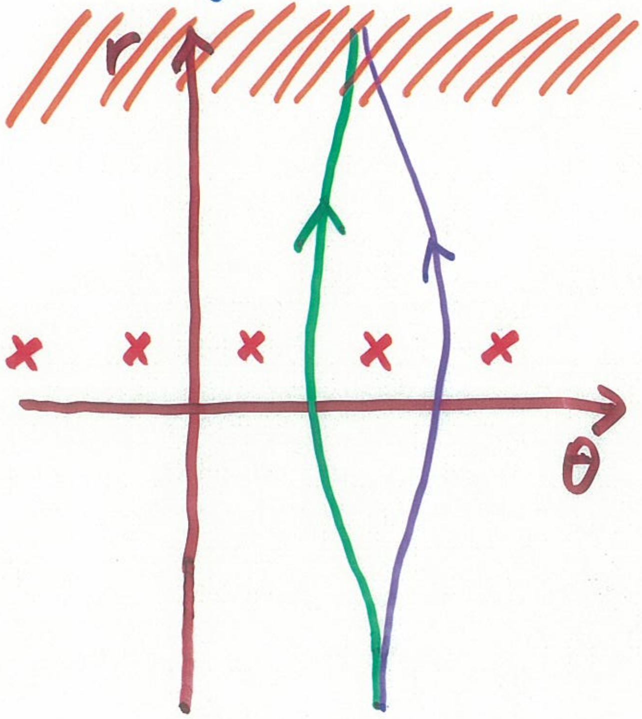
$\boxed{\widetilde{R}_0 = -\frac{6}{5}}$ $\quad -3 \quad -2 \quad -1 \quad 0 \quad 1$

$$\begin{array}{l}
\xrightarrow{x} \mathcal{O}(2)^5 \xrightarrow{x} \mathcal{O}(3) \to \mathcal{O}(7)^5 \to \\
\mathcal{O}(0)^{10} \xrightarrow{x} \mathcal{O}(1)^{10} \to \mathcal{O}(5)^{10} \to \mathcal{O}(6)^{10} \to \mathcal{O}(10)^{10} \to \quad \cdots \\
\cdots\ 0 \to \mathcal{O}(3) \xrightarrow{x} \mathcal{O}(4)^5 \to \mathcal{O}(8) \to \mathcal{O}(9)^5 \to \mathcal{O}(13) \to \\
\qquad \wr\wr \text{ q.is} \\
\cdots\ 0 \to \boxed{\Lambda^3 T^*_{\mathbb{P}^4}(1)} \to 0 \ \cdots
\end{array}$$

$\boxed{\widetilde{R}_0 = -\frac{8}{5}}$ $\quad -3 \quad -2 \quad -1 \quad 0 \quad 1$

$$\begin{array}{l}
\xrightarrow{x} \mathcal{O}(3)^5 \xrightarrow{x} \mathcal{O}(4) \xrightarrow{x^4} \mathcal{O}(8)^5 \to \\
\xrightarrow{x} \mathcal{O}(1)^{10} \xrightarrow{x} \mathcal{O}(2)^{10} \to \mathcal{O}(6)^{10} \to \mathcal{O}(7)^{10} \to \mathcal{O}(11)^{10} \to \quad \cdots \\
0 \to \mathcal{O}(0)^5 \xrightarrow{x^4} \mathcal{O}(4) \to \mathcal{O}(5)^5 \to \mathcal{O}(9) \to \mathcal{O}(10)^5 \to \mathcal{O}(14) \to \\
\quad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \wr\wr \text{ q.is} \\
0 \to \mathcal{O}(0)^5 \xrightarrow{x} \mathcal{O}(1)^{10} \xrightarrow{x} \mathcal{O}(2)^{10} \xrightarrow{x} \mathcal{O}(3)^5 \xrightarrow{x} \mathcal{O}(4) \to 0 \to \\
\quad \uparrow \qquad \wr\wr \text{ q.is} \\
0 \to \mathcal{O}(-1) \to 0 \\
\qquad \cong \Lambda^4 T^*_{\mathbb{P}^4}(1)
\end{array}$$

Change of window



Different windows

$\leftrightarrow$ Different $\mathcal{I}$'s

$\leftrightarrow$ Different equivalences of $MF_{\mathbb{Z}_N}(G)$ & $D^b(\mathrm{Coh}\, X_G)$

$$MF_{\mathbb{C}^\times}(PG(x))$$

$$\cup \qquad \cup$$

$$\mathcal{I}_1 \neq \mathcal{I}_2$$

$$MF_{\mathbb{Z}_N}(G(x)) \qquad\qquad D^b(\mathrm{Coh}\, X_G)$$

Difference: **Conifold monodromy**