” Liouville Theory, Modular Forms, and Elliptic Genera”

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When a CY manifold is non-compact or singular, a CFT describing string propagation on such a manifold has continuous as well as discrete representations. These CFT’s have a central charge above the threshold and are of non-minimal type. Let us call them generically as Liouville type theories. Since continuous and discrete representations mix under modular transformation, representations of Liouville theories do not have good modular behaviors. Then it is non-trivial to construct suitable modular invariants describing the geometry of non-compact CY such as elliptic genera. This work proposes a way of constructing elliptic genera for non-compact CY manifolds like ALE spaces and is an attempt at generalization of CY/LG corresondence for non-comact CY manifolds. It turns out that consistency of our approach hinges on some non-trivial identities of theta functions which have recently been proved mathematically by D.Zagier.

We start from the case of bosonic Liouville theory.

♠ $\mathcal{N} = 0$ Liouville Theory

In the case of bosonic Liouville the stress tensor is given by

$$T(z) = \frac{-1}{2}(\partial\phi)^2 + \frac{\mathcal{Q}}{2}\partial^2\phi$$

where $\mathcal{Q}$ is the background charge. Central charge is given by

$$c = 1 + 3\mathcal{Q}^2$$

There exist two types of representations in bosonic Liouville theory:

continuous reps.; $p > 0$

$$\chi_p(\tau) = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}(1-q^n)} = \frac{q^{\frac{p^2}{2}}}{\eta(\tau)}, \quad h = \frac{p^2}{2} + \frac{Q^2}{8}$$

$$\chi_p(-\frac{1}{\tau}) = \int_0^\infty dp' \cos(2\pi p p') \chi_{p'}(\tau),$$

identity rep.; $h = 0$

$$\chi_{h=0}(\tau) = \frac{q^{\frac{-Q^2}{8}}(1-q)}{\eta(\tau)},$$

$$\chi_{h=0}(-\frac{1}{\tau}) = \int_0^\infty dp \sinh(2\pi bp) \sinh(\frac{2\pi p}{b})\chi_p(\tau)$$

Let us introduce ZZ and FZZT branes and identify the character functions as the inner product

$$\chi_{h=0}(-\frac{1}{\tau}) = \langle ZZ|e^{i\pi\tau H^{(c)}}|ZZ\rangle$$
$$\chi_p(-\frac{1}{\tau}) = \langle FZZT; p|e^{i\pi\tau H^{(c)}}|ZZ\rangle$$

where $H^{(c)} = (L_0 + \bar{L}_0)/2$ is the closed string Hamiltonian. Thus we identify LHS as the open and RHS as closed string channel. We then find

Spectrum:

open	closed
$\left\{ \begin{array}{l} \text{continous rep.} \\ \text{identity rep.} \end{array} \right.$	continous rep.

There is no identity representation in closed string channel. This is consistent with the presence of mass gap and decoupling of gravity

$$h(e^{\alpha\phi}) = -\frac{\alpha^2}{2} + \frac{\alpha \mathcal{Q}}{2} = -\frac{(\alpha - \frac{1}{2}\mathcal{Q})^2}{2} + \frac{\mathcal{Q}^2}{8}$$
$$= \frac{p^2}{2} + \frac{\mathcal{Q}^2}{8} \geq \frac{\mathcal{Q}^2}{8} \text{ for } \alpha = ip + \frac{1}{2}\mathcal{Q}$$

If we use Ishibashi states $|p\rangle\rangle$ with momentum p

$$\langle\langle p|e^{i\pi\tau H^{(c)}}|p'\rangle\rangle = \delta(p - p')\chi_p(\tau)$$

boundary states are expanded as

$$|ZZ\rangle = \int_0^\infty dp \Psi_0(p)|p\rangle\rangle$$

$$|FZZT;p\rangle = \int_0^\infty dp' \Psi_p(p')|p'\rangle\rangle$$

We then have

$$|\Psi_0(p)|^2 = \sinh 2\pi pb \sinh \frac{2\pi p}{b}$$
$$\Psi_p(p')^* \Psi_0(p') = \cos 2\pi pp'$$

Solving these relations one finds the boundary wave-functions

$$\Psi_0(p) = \frac{2\pi i p \mu^{\frac{ip}{b}}}{\Gamma(1 + i2pb)\Gamma(1 + \frac{i2p}{b})}$$

$$\Psi_p(p') = \frac{-\mu^{\frac{ip'}{b}}}{2\pi i p'}\Gamma(1 - 2ibp')\Gamma(1 - \frac{2ip'}{b}) \times \cos(2\pi p p')$$

In order to apply to the string theory we consider supersymmetric version of Liouville theory

♠ $\mathcal{N} = 2$ Liouville Theory

There are two bosons (one of them is coupled to background charge and the other is a compact boson) and two free fermions in the system. It is known that $\mathcal{N} = 2$ Liouville theory is T-dual to $SL(2; R)/U(1)$ supercoset theory which describes 2 dim. black hole. In general $\mathcal{N} = 2$ Liouville geometrically is interpreted as describing the radial direction of a complex cone.

For the sake of simplicity consider the case with central charge

$$\hat{c} = \frac{c}{3} = 1 + \frac{2}{N}$$

which is T-dual to 2-dim black hole with an asymptotic radius of the cigar $\sqrt{2N}$.

Unitary representations of $\mathcal{N}=2$ algebra with $\hat{c}=1+\dfrac{2}{N}$

$$
\begin{cases}
\text{identity rep.} & j=0 & \text{vacuum} \\
\text{continous rep.} & j=\dfrac{1}{2}+i\dfrac{p}{Q} & \text{non-BPS states} \\
\text{discrete reps.} & j=\dfrac{s}{2} \quad 1\le s\le N & \text{BPS states, chiral primaries}
\end{cases}
$$

These representations are in one to one correspondence with those of $SL(2;R)/U(1)$ coset theory with level $k = N$.

We consider the sum over spectral flows of each $\mathcal{N} = 2$ representation in order to apply for string theory.

$$\chi_*(r;\tau,z) = \sum_{n \in r+NZ} q^{\frac{\hat{c}}{2}n^2} e^{2\pi i \hat{c} z n} ch_*(\tau, z + n\tau),$$

$$\hat{c} = c/3$$

and we obtain characters

1. Identity representations :

$$\chi_{id}(r;\tau); \quad r \in Z_N,$$

2. Continuous representations :

$$\chi_{cont}(p,m;\tau); \quad p \geq 0, \quad m \in Z_{2N},$$

$$h = p^2/2 + (m^2+1)/4N, Q = m/N$$

3. Discrete representations :

$$\chi_{dis}(s,r;\tau); \quad r \in Z_N, 1 \leq s \leq N, \ Q = s/N$$

S tranformations of these characters has the pattern

$$\text{(continous rep)} \xrightarrow{S} \text{(continous rep)}$$

$$\text{(discrete rep)} \xrightarrow{S} \text{(continous rep)} + \text{(discrete rep)}$$

Such a pattern was first observed in $\mathcal{N} = 4$ rep theory.

1. There are no identity reps in the RHS of above formulas.

2. It is still possible to show that $S^2 = C$.

We have three types of boundary states of $\mathcal{N} = 2$ theory corresponding to each representation. Their boundary wave functions are again given by the elements of the modular S matrix. We can compare our expressions with known results of $SL(2;R)/U(1)$ theory obtained by semi-classical method using DBI action. We reproduce wave functions of $D0, D1, D2$ branes of 2d black hole (Ribault-Schomerus). Thus the representation theory seems to work fairly well. However, the character formula themselves do not have good modular properties and it is non-trivial to construct conformal blocks with good modular behaviors.

♣ Geometry of $\mathcal{N} = 2$ Liouville fields

We consider models of the following type:

$\mathcal{N} = 2$ Liouville theory $\otimes$ $\mathcal{N} = 2$ minimal model

$$\boldsymbol{L_N \otimes M_k}$$

If we choose

$$\boldsymbol{N = k + 2}$$

the central charge becomes

$$\boldsymbol{c_L + c_M = 3(1 + \frac{2}{N}) + 3(1 - \frac{2}{k+2}) = 6}$$

and the theory describes (complex) 2 dimensional non-compact CY manifolds, i.e. ALE spaces. At $N = 1$ (without minima model), we have $\hat{c} = 3$ and the space-time of a conifold.

We may also consider the tensor products of Liouville theories and minimal models. These describe other various singular geometries like A_{N-1} spaces fibered on P^1 etc.

Elliptic genus:

$$Z(\tau; z) = Tr_{R\otimes R}(-1)^{F_L+F_R} e^{2\pi i z J_o^L} q^{L_0} \bar{q}^{\bar{L}_0}$$

$$Z(\tau; z=0) = \chi, \qquad \text{Euler number}$$

$$Z(\tau; z=1/2) = \sigma + ..., \quad \text{signature}$$

$$Z(\tau; z=(\tau+1)/2) = \hat{A} q^{-1/4} + ..., \ \hat{A} \text{ genus}$$

Elliptic genus is an invariant under smooth variation of parameters of the theory and is useful, for instance, in counting the number of BPS states. We compute the elliptic genus of a non-compact CY manifolds by pairing the Liouville theory with $\mathcal{N}=2$ minimal models.

CY/LG correspondence

We first recall the results of CY/LG correspondence. We consider a LG theory with a superpotential

$$W = g(X^{k+2} + Y^2 + Z^2)$$

which in the IR acquires scale invariance and reproduces the $\mathcal{N} = 2$ minimal theory with $\hat{c} = 1 - \frac{2}{k}$.

In the minimal theory the contribution to elliptic genus comes from the Ramond ground states

$$Z_{minimal}(\tau, z) = \sum_{\ell=0}^{N-2} ch^{\tilde{R}}_{\ell,\ell+1}(\tau; z)$$

On the other hand as the coupling parameter is turned off $g \to 0$, LG theory theory becomes a free theory of chiral field with $U(1)_R$ charge $= 1/N$. So that there is a free boson of charge $1/N$ and free fermion with charge $1/N - 1$. Combining these contributions one obtains

$$Z_{LG}(\tau, z) = \frac{\theta_1(\tau; (1-\frac{1}{N})z)}{\theta_1(\tau; \frac{1}{N}z)}$$

and in fact these two agree with each other

$$Z_{minimal} = Z_{LG}$$

Witten

We want to try a similar construction in Liouville sector. Ramond ground states are given by

$$\chi_{dis}(s, s-1; \tau, z), \quad s = 1, \cdots, N$$

$$Z_{Liouville} = \sum_{s=1}^{N} \chi_{dis}^{\tilde{R}}(s, s-1; \tau, z)$$
$$= \mathcal{K}_{2N}(\tau, \frac{z}{N}) \frac{\theta_1(\tau, z)}{\eta(\tau)^3}$$

Here $\mathcal{K}_\ell$ denotes the Appell function

$$\mathcal{K}_{\ell}(\tau, z) = \sum_{m} \frac{q^{\frac{\ell}{2}m^2} e^{2\pi i m \ell z}}{1 - e^{2\pi i z} q^m}$$

Unlike the theta functions of minimal model, Appell function in Liouville theory does not have a good modular transformation law.

When we couple minimal and Liouville theory to compute elliptic genera of A_{N-1} spaces, we may use the orbifoldization procedure and we find

$$
\begin{aligned}
&Z_{ALE(A_{N-1})}(\tau, z) \\
&= \frac{1}{N} \sum_{a,b \in Z_N} q^{a^2} e^{4\pi i a z} Z_{minimal}(\tau, z + a\tau + b) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times Z_{Liou}(\tau, z + a\tau + b)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N} \sum_{a,b \in \mathrm{Z_N}} q^{\frac{a^2}{2}} e^{2\pi i a z} (-1)^{a+b} \frac{\theta_1(\tau; \frac{N-1}{N}(z + a\tau + b))}{\theta_1(\tau; \frac{1}{N}(z + a\tau + b))} \\
&\qquad\qquad \times \mathcal{K}_{2N}(\tau, \frac{1}{N}(z + a\tau + b)) \frac{\theta_1(\tau; z)}{\eta(\tau)^3}
\end{aligned}
$$

In the special case of $N = 2$ we have ($y = e^{2\pi i z}$)

$$Z_{ALE(A_1)}(\tau; z) = \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} y^{n+\frac{1}{2}}}{1 - yq^n} \frac{\theta_1(\tau; z)}{\eta(\tau)^3}$$

(This coincides with a massless character of $\mathcal{N} = 4$ algebra.)

Unfortunately these formulas do not have well-behaved modular properties and we must make a suitable modification. Elliptic genus is associated with a CFT defined on the torus and hence it must be invariant under $SL(2; Z)$ or under one of its subgroups. Since we are dealing with SCFT, it seems natural to demand invariance under the subgroup $\Gamma(2)$ which leave fixed the spin structures.

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathrm{Z}),\ a = d = 1, b = c = 0 \mod 2 \right\}$$

K3 Elliptic Genus

A hint comes from the study of elliptic genus of K3 surface

$$Z_{K3}(\tau, z) = 8 \left[\left(\frac{\theta_3(\tau, z)}{\theta_3(\tau)} \right)^2 + \left(\frac{\theta_4(\tau, z)}{\theta_4(\tau)} \right)^2 + \left(\frac{\theta_2(\tau, z)}{\theta_2(\tau)} \right)^2 \right]$$

This formua can be easily derived by orbifold calculation on T^4/Z_2 or we may use LG theory and LG/CY correspondence.

We can check $Z_{K3}(z=0)=24, Z_{K3}(z=1/2)=16+\ldots, Z_{K3}(z=\tau/2)=-2q^{-1/4}+\ldots$.

In this case we can use the $\mathcal{N}=4$ representation theory. At $\hat{c}=2$ $\mathcal{N}=4$ theory contains $SU(2)$ symmetry at level 1.

Unitary representations in R sector are

massive rep. : $ch^{\tilde{R}}(h;\tau,z)=q^{h-\frac{1}{8}}\dfrac{\theta_1(\tau,z)^2}{\eta(\tau)^3}$

massless rep. : $ch_0^{\tilde{R}}(I=0;\tau,z),\ ch_0^{\tilde{R}}(I=1/2;\tau,z)$

Relation : $ch_0^{\tilde{R}}(I=1/2)+2ch_0^{\tilde{R}}(I=0)=-q^{-\frac{1}{8}}\dfrac{\theta_1^2}{\eta^3}$

Under spectral flow

$$R : I = 0 \iff NS : I = 1/2$$
$$R : I = 1/2 \iff NS : I = 0$$

Decomposition formula

$$\begin{aligned} ch_0^{\tilde{R}}(I=0,\tau,z) &= \left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2 - h_3(\tau)\left(\frac{\theta_1(z)}{\eta(\tau)}\right)^2 \\ &= \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2 - h_4(\tau)\left(\frac{\theta_1(z)}{\eta(\tau)}\right)^2 \\ &= \left(\frac{\theta_2(z)}{\theta_2(0)}\right)^2 - h_2(\tau)\left(\frac{\theta_1(z)}{\eta(\tau)}\right)^2 \end{aligned}$$

where

$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(\tau)} \sum_m \frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}$$

$$h_4(\tau) = \frac{1}{\eta(\tau)\theta_4(\tau)} \sum_m \frac{q^{m^2/2-1/8}(-1)^m}{1-q^{m-1/2}}$$

$$h_2(\tau) = \frac{1}{\eta(\tau)\theta_2(\tau)} \sum_m \frac{q^{m^2/2+m/2}}{1+q^m}$$

Rewrite K3 genus as

$$
\begin{aligned}
Z_{K3}(z) &= 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2+\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau)}\right)^2\right] \\
&= 24ch_0^{\tilde{R}}(I=0;z)+8\sum_{i=2,3,4} h_i(\tau)\frac{\theta_1(z)^2}{\eta(\tau)^2}
\end{aligned}
$$

We note that

$$
8\,\eta(\tau)\sum_{i=2,3,4} h_i(\tau)=2q^{-1/8}\left[1-\sum_{n=1} a_n q^n\right]
$$

where integer coefficients a_n are all positive. Thus we further rewrite

$$Z_{K3}(z) = 20ch_0^{\tilde{R}}(I=0;z) - 2ch_0^{\tilde{R}}(I=1/2;z) - \sum_{n=1} a_n q^{n-1/8} \frac{\theta_1(z)^2}{\eta(\tau)^3}$$

or

$$Z_{K3}(z') = 20ch_0^{NS}(I=1/2;z) - 2ch_0^{NS}(I=0;z) + \sum_{n=1} a_n q^{n-1/8} \frac{\theta_3(z)^2}{\eta(\tau)^3}$$

$$z' = z - (\tau/2 + 1/2)$$

We see the theory contains

$1 \quad I = 0$ rep.

$20 \quad I = 1/2$ reps.

∞ of massive reps. $(h = 1, 2, \cdots)$

Seiberg

$I = 0$ rep. corresponds to the gravity multiplet and $I = 1/2$ rep. to matter multiplets (vector in IIA, tensor in IIB) in SUGRA description. By throwing away the gravity multiplet we can decompactify K3 into a sum of ALE spaces; it is known K3 may be decomposed into a sum of 16 A_1 spaces. Decompactification corresponds to dropping $I = 0$ massless representation or $h = 0$ massive representation. This suggests

$$Z_{K3,decompactified} = 8\left[\left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2 + \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2\right]$$

Thus we propose

$$Z_{A_1}(z) = \frac{1}{2}\left[\left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2 + \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2\right]$$

Note that

$$Z_{A_1}(z) = ch_0^{\tilde{R}}(I = 0; z) + \frac{1}{2}\eta(\tau)\left(h_3(\tau) + h_4(\tau)\right)\frac{\theta_1(z)^2}{\eta(\tau)^3}$$

where again the expansion

$$\frac{1}{2}\eta(\tau)\sum_{i=3,4} h_i(\tau) = \sum_{n=1} b_n q^n$$

has all positive integer coefficients b_n.

We also propose

$$Z_{A_{N-1}}(z) = (N-1)\frac{1}{2}\left[\left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2 + \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2\right]$$

Above construction of Z_{A_1} suggests that instead of using the irreducible character $ch_0^{\tilde{R}}$ we should use a combination with massive reps.

$$ch_0^{\tilde{R}}(I=0;z)+\frac{1}{2}\eta(\tau)(h_3(\tau)+h_4(\tau))\frac{\theta_1(z)^2}{\eta(\tau)^3}$$
$$=\frac{1}{2}\left[\left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2+\left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2\right]$$

which has a good modular property. We call this combination as the topological part of the massless representation and consider it as a conformal block in non-compact CFT.

Let us now go back to the orbifold formula for Z_{ALE} and replace the Appell function by its topological part. We first use the decomposition of level k N=4 massless character

$$ch_0^{\tilde{R}}(k; I=0; \tau, z) = \frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2k} + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2k}\right]$$
$$+ \text{ massive reps.}$$

Then use its relation to Appell function

$$ch_0^{\tilde{R}}(k; I=0; \tau, z) = \frac{2\theta_1(\tau,z)^2}{\eta(\tau)^3\theta_1(\tau,2z)}\mathcal{K}_{2(k+1)}(\tau,z)$$

By combining these we find

$$[\mathcal{K}_{2N}(\tau,z)]_{top} = \frac{\eta(\tau)^3}{4}\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)^2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^{2(N-1)}\right]$$

Then the orbifold formula predicts

$$Z_{A_{N-1}}(\tau, z) = \frac{1}{4N} \sum_{a,b} q^{\frac{a^2}{2}} e^{2\pi i a z} (-1)^{a+b} \frac{\theta_1(\tau; \frac{N-1}{N} z_{a,b})}{\theta_1(\tau; \frac{1}{N} z_{a,b})^3}$$
$$\times \theta_1(\tau, \frac{2}{N} z_{a,b}) \left[\left(\frac{\theta_3(\tau, \frac{1}{N} z_{a,b}}{\theta_3(\tau)} \right)^{2(N-1)} + \left(\frac{\theta_4(\tau, \frac{1}{N} z_{a,b})}{\theta_4(\tau)} \right)^{2(N-1)} \right]$$
$$\times \theta_1(\tau, z)$$

where $z_{a,b} = z + a\tau + b$.

Strikingly, RHS of this formula agrees exactly with the proposed expression for $Z_{A_{N-1}}$

$$Z_{A_{N-1}}(\tau, z) = (N-1)\frac{1}{2}\left[\left(\frac{\theta_3(z)}{\theta_3(0)}\right)^2 + \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^2\right]$$

Actually this is a special case of non-trivial identities of theta functions

$$\frac{1}{2N}\sum_{a,b} q^{\frac{a^2}{2}} e^{2\pi i a z}(-1)^{a+b}\frac{\theta_1(\tau;\frac{N-1}{N}z_{a,b})\theta_1(\tau,\frac{2}{N}z_{a,b})\theta_1(\tau;z)}{\theta_1(\tau;\frac{1}{N}z_{a,b})^3}$$

$$\times\left(\frac{\theta_i(\tau,\frac{1}{N}z_{a,b})}{\theta_i(\tau)}\right)^{2(N-1)}$$

$$= (N-1)\left(\frac{\theta_i(z)}{\theta_i(\tau)}\right)^2, \; i = 2, 3, 4$$

Identities for $i = 2, 3, 4$ transform into each other under $SL(2; \mathrm{Z})/\Gamma(2)$.

It is easy to show these for $N = 2$ by addition formula of theta functions and we have checked their validity by Maple for lower values of N.

Recently an elegant mathematical proof of these identities for general N has been found by D.Zagier using the method of residue integrals.

♣ Summary

When a CY manifold is non-compact, string theory is described by a CFT possessing continuous as well as discrete representations. Characters of representations of such CFT transform in a peculiar manner under S transformation

$$\text{discrete} \underset{S}{\longrightarrow} \sum \text{discrete} + \int \text{continuous}$$

$$\text{continuous} \underset{S}{\longrightarrow} \int \text{continuous}$$

Mathematical nature of such transformation is currently not well understood. We found an empirical rule in constructing conformal blocks which have good modular behavior and obtained elliptic genera of some non-compact CY manifolds.

We have yet to figure out a simple LG interpretation of the Liouville part of elliptic genera

$$Z_{Liouville} = \frac{1}{4}\frac{\theta_1(\tau, z)\theta_1(\tau, \frac{2}{N}z)}{\theta_1(\tau; \frac{1}{N}z\ )^2}$$

$$\times \left[\left(\frac{\theta_3(\tau, \frac{1}{N}z)}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau, \frac{1}{N}z)}{\theta_4(\tau)}\right)^{2(N-1)}\right]$$