# Ginsparg-Wilson relation and Chiral Gauge Theories on the lattice with exact gauge invariance

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Plan of this talk:

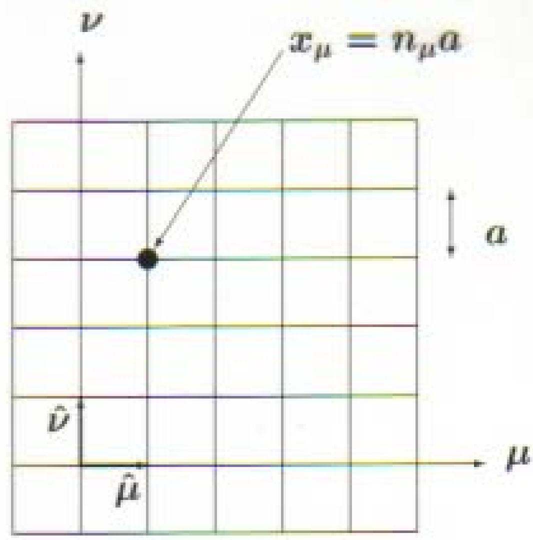
1. Exact chiral symmetry on the lattice
   - Ginsparg-Wilson relation and its gauge-covariant solution
   - Index theorem on the lattice
   - Weyl fermions on the lattice and topological properties of gauge anomaly
2. Numerical implementation of U(1) chiral gauge theories with exact gauge invariance
3. Admissibility-$\epsilon$ expansion in local cohomology problem of Non-abelian anomaly

Based on


- D. Kadoh, Y. Kikukawa and Y. Nakayama, "Solving the local cohomology problem in U(1) chiral gauge theories within a finite lattice", hep-lat/0309067
- D. Kadoh and Y. Kikukawa, "A numerical solution to the local cohomology problem in U(1) chiral gauge theories", in preparation
- D. Kadoh and Y. Kikukawa, "Field tensor-based cohomological analysis of the axial anomaly in abelian lattice gauge theories", in preparation

## Lattice Gauge Theory

$\psi(x) \qquad x_\mu = n_\mu a \; (n_\mu \in \mathbb{Z}^4)$




The differential of the field can be replaced by difference:

$$\partial_\mu \psi(x) = \frac{1}{a}\left(\delta_{x+\hat{\mu},y} - \delta_{x,y}\right)\psi(y) = \frac{1}{a}\left(\psi(x+\hat{\mu}) - \psi(x)\right)$$

### Gauge invariance on the lattice

- Link variable and its gauge transformation:

$$U_\mu(x) = \mathrm{e}^{iaA_\mu(x)} \in G, \quad U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat{\mu})$$

- Gauge-covariant difference operator:

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat{\mu}) - \psi(x)\right)$$

- Field strength of lattice gauge field

$$[\nabla_\nu, \nabla_\mu]\,\psi(x) = (1 - P_{\mu\nu}(x))\; U_\nu(x)U_\mu(x+\hat{\nu})\psi(x+\hat{\mu}+\hat{\nu})$$

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\nu(x+\hat{\nu})^{-1}U_\nu(x)^{-1}$$

- Gauge-invariant action of lattice gauge field

$$S = \frac{1}{g^2}\sum_{x,\mu\nu} \mathrm{Re}\,\mathrm{Tr}\,(1 - P_{\mu\nu}(x))$$

- Invariant Path-integral measure

$$\prod_{x,\mu} dU_\mu(x)$$

## Current topics and activities in lattice gauge theories (list, not complete)

- Lattice QCD
  - **Simulation of full QCD including "light" three flavor quarks**
    Tsukuba-KEK, Riken-BNL, ···
    $m_{PS}/m_V \lesssim \mathbf{0.6} \gg \mathbf{0.18}$,
    Chiral limit extrapolation ⇔ Chiral perturbation theory !?
    (Theoretical issues about Hadronic decays)
  - **Heavy quark physics (c,b)** ⇔ B-factories
  - **QCD phase transition at Finite $T$ and $\mu$**
    various phases; Quark-Gluon plasma, Color superconductivity
    Issues about a finite $\mu$ : complex action
    * reweighting, Tayler expansion in $\mu$ ($\mu \lesssim T_c$) Ejiri
    * imaginary $\mu$
    * factorization Nishimura, Ambjorn et al.

- Symmetries in lattice gauge theories
  - **Exact chiral symmetry**
    * Ginsparg-Wilson relation
    * Neuberger's overlap Dirac operator
      application to QCD
      construction of chiral gauge theories
  - **Exact supersymmetry**
    various proposals:
    * Ito, Kato, Sawanaka, So, Ukita
    * Kaplan, Katz, Unsal, Cohen ( ⇔ Matrix model)
    * Sugino

- Other topics

## Chiral symmetry on the lattice

- Species doubling problem / Nielsen-Ninomiya theorem
- **Ginsparg-Wilson relation**: Condition of the chiral limit for lattice fermions

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

In terms of the fermion propagator $S_F = D^{-1}$:

$$\gamma_5 S_F(x,y) + S_F(x,y)\gamma_5 = \gamma_5 2\delta(x,y)$$

  - derived by block spin transformation (the IR fixed point of free fermion)
  - chiral symmetry is broken only in local contact terms

- **Neuberger's lattice Dirac operator** Neuberger, Neuberger-Y.K.
  (Gauge-covariant solution to the GW relation)

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right) = \frac{1}{2a}\left(1 + \gamma_5\frac{H}{\sqrt{H^2}}\right)$$

$$X = \left(D_{\rm w} - \frac{m_0}{a}\right), \quad H = \gamma_5 X, \quad (0 < m_0 < 2)$$

$$D_{\rm w} = \sum_\mu \left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\left(\nabla_\mu \nabla_\mu^\dagger\right)\right\}$$

- **GW rel. implies an exact symmetry of the fermion action!** Lüscher
  Under the following transformation,

$$\delta\psi(x) = \gamma_5\left(1 - 2aD\right)\psi(x), \quad \delta\bar\psi(x) = \bar\psi(x)\gamma_5$$

the fermion action is invariant,

$$\delta S = a^4 \sum_x \bar\psi\left\{D\gamma_5\left(1-2aD\right) + \gamma_5 D\right\}\psi(x) = 0$$

  - local transformation as long as $D$ is local
    note: locality and analyticity

$$\|D(x-y)\| < Ce^{-|x-y|/\varrho}$$

$$\Longleftarrow \quad \frac{\partial^l}{\partial k^l}\tilde D(k) = \sum_x e^{ikx}(ix)^l D(x) < \infty$$

- Chiral anomaly

  Chiral transformation depends on gauge fields

$$\delta\psi(x) = \gamma_5\,(1-2aD)\psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_5$$

Chiral Jacobian

$$\delta\left[\prod_x d\psi(x)\, d\bar{\psi}(x)\right] = \left[\prod_x d\psi(x)\, d\bar{\psi}(x)\right](-2)\mathrm{Tr}\gamma_5(1-aD)$$

Chiral anomaly in the (classical) continuum limit

Yamada-Y.K., Fujikawa, Suzuki, Adams

$$-\frac{2}{a^4}\mathrm{tr}\gamma_5(1-aD)(x,x) = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}F^a_{\mu\nu}(x)F^a_{\lambda\rho}(x) + \mathcal{O}(a)$$

- **Weyl Fermions on the Lattice** Narayanan-Neuberger, Niedermayer, Lüscher

$$\hat{\gamma}_5 \equiv \gamma_5\,(1-2aD)\,, \quad \{\hat{\gamma}_5\}^2 = 1$$

$$\gamma_5 D + D\hat{\gamma}_5 = 0$$

Weyl fermion:

$$\left(\frac{1-\hat{\gamma}_5}{2}\right)\psi_L(x) = \psi_L(x), \qquad \bar{\psi}_L(x)\left(\frac{1+\gamma_5}{2}\right) = \bar{\psi}_L(x)$$

  – Fermion number violation due to anomaly/ topologically non-trivial gauge fields

- Zero modes of $D$ are chiral eigenstates !

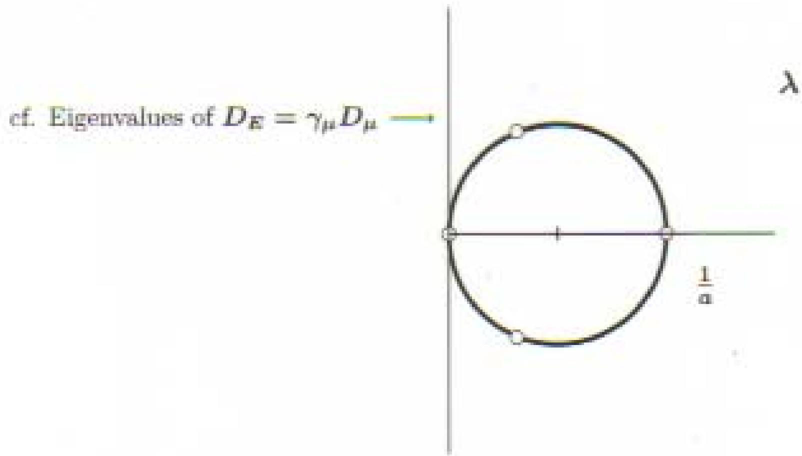
$$\begin{aligned} D\psi_0(x) &= 0 \\ D\gamma_5\psi_0(x) &= (-\gamma_5 D + 2aD\gamma_5 D)\,\psi_0 = 0 \end{aligned}$$

- Eigenvalues of $D$

$$D + D^\dagger = 2aD^\dagger D = 2aDD^\dagger \quad \text{(normal)}$$
$$D^\dagger = \gamma_5 D \gamma_5 \quad (\gamma_5\text{-conjugate})$$

$$\begin{aligned} \lambda + \lambda^* - 2a\lambda^*\lambda &= (-2a)\left[(\lambda - 1/2a)(\lambda - 1/2a)^* - (1/2a)^2\right] \\ &= 0 \end{aligned}$$

$\therefore$ on the circle with the radius $1/2a$ ( center $(1/2a, 0)$ )




$$\begin{aligned} \lambda = 0 \quad &: \quad \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) \qquad n_\pm \\ \lambda = 1/a \quad &: \quad \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) \qquad N_\pm \\ \lambda \neq 0, 1/a &: \quad \text{pair-wise} \begin{cases} \lambda \to \psi_\lambda \\ \lambda^* \to \gamma_5\psi_\lambda \end{cases} \quad \psi_\lambda^\dagger\gamma_5\psi_\lambda = 0 \end{aligned}$$

- Index theorem

$$\begin{aligned} \mathrm{Tr}\,\{\gamma_5(1 - aD)\} &= \sum_\lambda \psi_\lambda^\dagger\gamma_5\psi_\lambda - a\sum_\lambda \lambda\,\psi_\lambda^\dagger\gamma_5\psi_\lambda \\ &= \sum_{\lambda=0,1/a} \psi_\lambda^\dagger\gamma_5\psi_\lambda - a\sum_{\lambda=1/a} \frac{1}{a}\,\psi_\lambda^\dagger\gamma_5\psi_\lambda \\ &= (n_+ - n_-) + (N_+ - N_-) - (N_+ - N_-) \\ &= n_+ - n_- \end{aligned}$$

$2N_f\,\mathrm{Index}(D) = -2\mathrm{Tr}\gamma_5(1 - aD)$ at a finite lattice spacing

Topological charge of lattice gauge fields $\{U_\mu(x)\}$

$$Q = \sum_x \operatorname{tr}\{\gamma_5(1 - aD)(x,x)\} = -\frac{1}{2}\sum_x \operatorname{tr}\left\{\frac{H}{\sqrt{H^2}}(x,x)\right\}$$

- depends on $\{U_\mu(x)\}$ smoothly and locally

  **Admissibility condition**

$$\| 1 - P_{\mu\nu}(x) \| < \epsilon, \qquad \epsilon < \frac{1}{30}\left(1 - |1 - m_0|^2\right)$$

then

$$\|H^2\| \geq \left\{(1 - 30\epsilon)^{\frac{1}{2}} - |1 - m_0|\right\}^2 > 0$$

- Space of lattice gauge fields $\{U_\mu(x)\}$ have non-trivial topological structure!
- A fermionic definition using Wilson-Dirac operator

  Narayanan-Neuberger(1995), Ito-Iwasaki-Yoshie(1987)

  cf. Geometrical definition Lüscher (1982), Pillips-Stone (1996)
- Topological property of chiral anomaly

  chiral anomaly / the density of topological charge:

$$q(x) = \operatorname{tr}\{\gamma_5(1 - aD)(x,x)\}$$

$$\sum_x \delta q(x) = 0 \qquad (\Leftarrow \quad \hat{\gamma}_5^2 = 1)$$

  – $\Longrightarrow$ **Exact gauge invariance in chiral gauge theories**

## Weyl fermions on the lattice and gauge anomaly

Weyl fermion:

$$\left(\frac{1-\hat{\gamma}_5}{2}\right)\psi_L(x)=\psi_L(x), \qquad \bar{\psi}_L(x)\left(\frac{1+\gamma_5}{2}\right)=\bar{\psi}_L(x)$$

Path-Integral measure $\Longleftrightarrow$ Chiral bases: $\{v_i(x)\}$ and $\{\bar{v}_k(x)\}$

$$\hat{P}_- v_i(x)=v_i(x), \qquad \hat{P}_-=\left(\frac{1-\hat{\gamma}_5}{2}\right)$$

$$\bar{v}_k(x)P_+=\bar{v}_k(x), \qquad P_+=\left(\frac{1+\gamma_5}{2}\right)$$

$$\psi_L(x)=\sum_i v_i(x)c_i, \quad \bar{\psi}_L(x)=\sum_k \bar{c}_k\bar{v}_k(x)$$

$$\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L]\equiv\prod_i dc_i\prod_k d\bar{c}_k\,[U_\mu(x)]$$

Gauge anomaly:

- Chiral determinant or effective action — Overlap Formula

$$\begin{aligned} e^{\Gamma_{\rm eff}(U_\mu)} &= \int\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L]\;e^{-\sum_x\bar{\psi}_L(x)D\psi_L(x)} \\ &= \int\prod_i dc_i\prod_k d\bar{c}_k\,e^{-\sum_{k,i}\bar{c}_k(\bar{v}_k,Dv_i)c_i}=\det\left(\bar{v}_k D v_j\right) \end{aligned}$$

- Variation of effective action w.r.t. gauge field

$$U_\mu(x)\longrightarrow U_\mu(x)+\delta_\eta U_\mu(x), \qquad \delta_\eta U_\mu(x)=\eta_\mu(x)U_\mu(x)$$

$$\begin{aligned} \delta_\eta\Gamma_{\rm eff} &= \delta_\eta{\rm Tr\,Ln}\left(\bar{v}_k,Dv_i\right) \\ &= {\rm Tr}\left\{(\delta_\eta D)\hat{P}_-D^{-1}P_+\right\}+\sum_i(v_i,\delta_\eta v_i) \\ \therefore\;\sum_x -i\,\eta_\mu J_\mu &= {\rm Tr}\left\{(\delta_\eta D)\hat{P}_-D^{-1}P_+\right\}+\sum_i(v_i,\delta_\eta v_i) \end{aligned}$$

- Gauge anomaly $\qquad \eta_\mu(x)=-i\nabla_\mu\omega(x),\ \delta_\eta D=i[\omega,D]$

$$\begin{aligned} \sum_x\omega\cdot\nabla_\mu^* J_\mu &= i{\rm Tr}\omega\left(P_+-\hat{P}_-\right)+\sum_i(v_i,\delta_\eta v_i) \\ &= -i{\rm Tr}\omega\gamma_5\left(1-aD\right)+\sum_x\omega\cdot\nabla_\mu^* j_\mu \end{aligned}$$

## Topological property of gauge anomalies and the exact gauge invariance

- $U(1)$ case: the gauge anomaly is topological ! (cf. Index theorem)

$$q(x) = \mathrm{tr}\left\{\gamma_5(1-aD)(x,x)\right\}$$

$$\sum_x \delta q(x) = 0 \qquad \Leftarrow \qquad \delta\hat{\gamma}_5\hat{\gamma}_5 + \hat{\gamma}_5\delta\hat{\gamma}_5 = 0$$

then one can show non-perturbatively — Lüscher

$$q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu k_\mu(x)$$

- $F_{\mu\nu}(x) = \frac{1}{i}\ln P_{\mu\nu}(x)$
- $k_\mu(x)$ : local, gauge-invariant current
- For an anomaly-free multiples with $\sum_\alpha e_\alpha^3 = 0$,

$$\sum_\alpha e_\alpha q^\alpha(x) = \partial^*_\mu \bar{k}_\mu(x), \qquad \bar{k}_\mu(x) = \sum_\alpha e_\alpha\{k_\mu(x)\}|_{U\to U^{e_\alpha}}$$

- An exactly conserved gauge current (gauge-covariant(invariant))

$$J_\mu(x) \equiv i\mathrm{Tr}\left\{\left(\frac{\delta}{\delta\eta_\mu(x)}D\right)\hat{P}_- D^{-1}P_+\right\} - \bar{k}_\mu(x)$$

- Gauge-invariant measure (choice of chiral basis)

$$U^t_\mu(x) = \mathrm{e}^{itA_\mu(x)} \qquad t\in[0,1]$$


$$v_i(x) = \begin{cases} Q_1 v^0_1(x) W^{-1} & \text{if } i=1 \\ Q_1 v^0_i(x) & \text{otherwise} \end{cases}$$

- $Q_t$ : Evolution of the projection operator (chiral basis)

$$\partial_t Q_t = [\partial_t P_t, P_t]Q_t, \qquad Q_0 = 1$$
$$P_t = Q_t P_0 Q_t^{-1}, \qquad P_t = \hat{P}_-|_{U=U_t}$$

- $W$ : "Local counter term"

$$\ln W = i\int_0^1 dt \sum_x A_\mu(x)\left\{k_\mu(x)\right\}|_{A\to tA} - \int_0^1\int_0^1 dsdt\,\mathrm{Tr}\left\{\hat{P}_-[\partial_t\hat{P}_-, \partial_s\hat{P}_-]\right\}$$

## Numerical implemetation of $U(1)$ chiral gauge theory with exact gauge invariance

- How to compute $k_\mu(x)$ : cohomological analysis

1. Poincaré lemma on the lattice (the infinite lattice)

   **Lemma** Let $f$ be a $k$-form which satisfies

   $$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0 \ \text{ if } \ k = 0$$

   Then there exist a form $g \in \Omega_{k+1}$ such that

   $$f = d^* g$$

   – $f$ and $g$ have the same locality property

2. Vector potential representation of the link variables

   $$U_\mu(x) = \mathrm{e}^{iA_\mu(x)}, \qquad |A_\mu(x)| \leq \pi(1 + 4 \parallel x \parallel)$$
   $$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

   – One-parameter family of admissble gauge fields

   $$U^t_\mu(x) = \mathrm{e}^{itA_\mu(x)}, \ t \in [0, 1]$$

3. Topological property of $q(x)$

   $$q(x) = \sum_y j_\mu(x, y) A_\mu(y), \qquad j_\mu(x, y) = \int_0^1 dt \, \frac{\partial q(x)}{\partial A_\mu(y)}\bigg|_{A \to tA}$$

   $$\sum_x j_\mu(x, y) = 0, \qquad j_\mu(x, y)\overleftarrow{\partial}^*_\mu = 0$$

- Numerical computation of $k_\mu(x)$ on the finite lattice ($V = L^4, L^2$)

  - Poincaré lemma on the lattice (the finite lattice)

**Lemma a** Let $f$ be a $k$-form which satisfies

$$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0 \ \text{ if } \ k = 0$$

Then there exist a form $g \in \Omega_{k+1}$ and a form $\Delta f \in \Omega_k$ such that

$$f = d^* g + \Delta f, \qquad |\Delta f_{\mu_1 \cdots \mu_k}(x)| < c L^\sigma \mathrm{e}^{-L/2\varrho}$$

**Lemma b** Let $f$ be a $k$-form which satisfies

$$d^* f = 0 \ \text{ and } \ \sum_{x \in \Gamma_n} f(x) = 0$$

Then there exist a form $g \in \Omega_{k+1}$ such that

$$f = d^* g$$

* (a lattice counter part of the corollary of de Rham theorem)

  - Vector potential representation of the link variables on the finite lattice

$$U_\mu(x) = \mathrm{e}^{i \tilde{A}_\mu(x)} V_{[m]}(x, \mu)$$

$$F_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2}$$

$$\begin{cases} |\tilde{A}_\mu(x)| \leq \pi(1 + 4 \parallel x \parallel) & (\parallel x \parallel \leq L/2) \\ |\tilde{A}_\mu(x)| \leq \pi(1 + 2L + 2(n-1)L^2) & \text{(otherwise)} \end{cases}$$

* $\tilde{A}_\mu(x)$ is periodic and bounded !
* Magnetic fluxes: topological invariant of admissible $U(1)$ gauge fields

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t} F(x + s\hat{\mu} + t\hat{\nu})$$

– Numerical computation of the bi-local current $j_\mu(x,y)$

* Rational approximation

$$q(x) = -\frac{1}{2}\mathrm{tr}\left\{\frac{H_w}{\sqrt{H_w^2}}(x,x)\right\}$$
$$\fallingdotseq -\frac{1}{2}\mathrm{tr}\left\{h_w\sum_{k=1}^{N_r}\frac{b_k}{h_w^2+c_{2k-1}}(x,x)\right\}$$

· Optimized coefficients $c_k$ and $b_k$ (Zolotarev optimization)

$$\frac{\partial q(x)}{\partial \tilde{A}_\nu(y)} \fallingdotseq -\frac{1}{2}\mathrm{tr}\left\{\sum_{k=1}^{N_r} b_k\frac{1}{h_w^2+c_{2k-1}}\times \left(c_{2k-1}v_\mu(y)-h_w v_\mu(y)h_w\right)\frac{1}{h_w^2+c_{2k-1}}(x,x)\right\},$$

* Integration by Guassian Quadrature (Gauss-Legendre) formula

$$j_\nu(x,y)\fallingdotseq\sum_{i=1}^{N_g} w_i\left(\frac{\partial q(x)}{\partial\tilde{A}_\nu(y)}\right)_{\tilde{A}\to t_i\tilde{A}}$$

· $\{(t_i,w_i)|i=1,\cdots,N_g(\simeq 20)\}$ : the set of the absscissas and weights

$$\sum_x j_\mu(x,y)=O(10^{-15}),\qquad j_\mu(x,y)\overleftarrow{\partial_\mu^*}=0 \text{ (exact)}$$

– Result of the cohomological analysis in two dimensions

$$q(x)=q_{[m]}(x)+\gamma_{[m,w]}\epsilon_{\mu\nu}\tilde{F}_{\mu\nu}(x)+\partial_\mu^* h_\mu(x)+\Delta q(x)$$

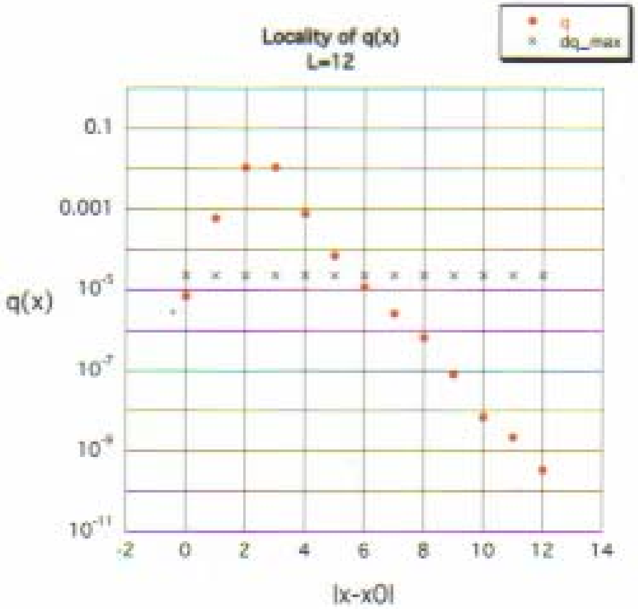
$$\bar{k}_\mu(x)\equiv\sum_\alpha e_\alpha\{h_\mu(x)+\Delta h_\mu(x)\}^\alpha+\Delta k_\mu(x)$$

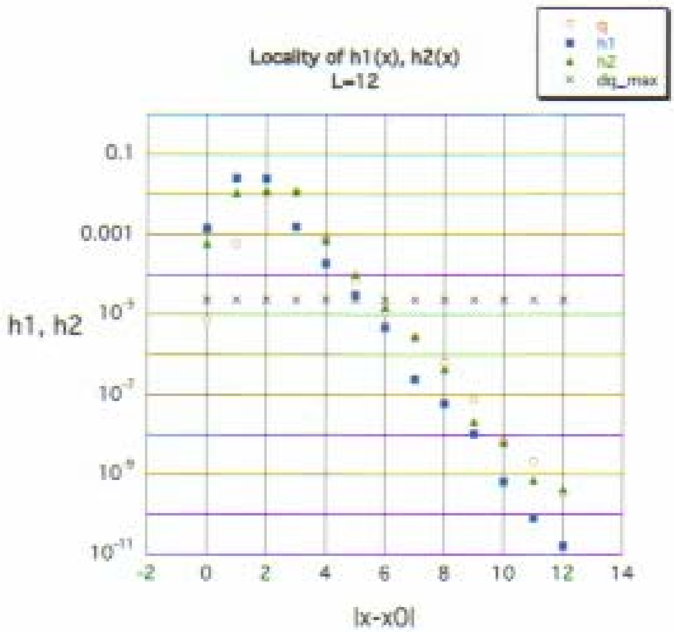
- Numerical solutions in two-dimensions

  Plots:

  apply a small variation: $\boldsymbol{\eta_\mu(x) = 0.05 \times 2\pi\, \delta_{x,x_0}\delta_{\mu,1}}$

$$\delta_\eta f(r) = \max\left\{ |\delta_\eta f(x)| \,\middle|\, r = \| x - x_0 \| \right\}$$



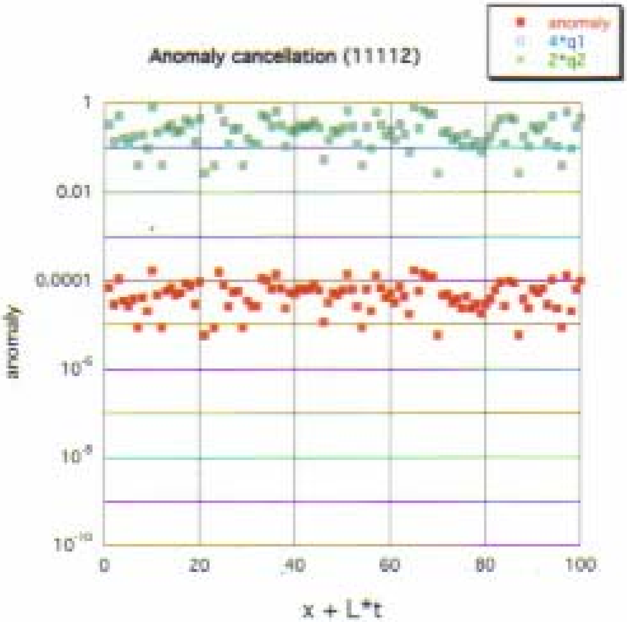



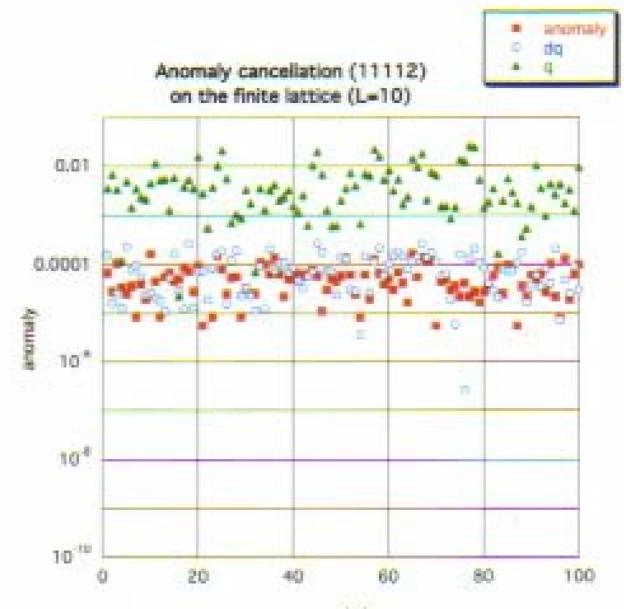
Plots:

anomaly part : $\mathcal{A}(x) = q_{[m]}(x) + \gamma_{[m,w]}\epsilon_{\mu\nu}\tilde{F}_{\mu\nu}(x)$

11112 model : $\sum_{i=1}^{4} e^2 - (2e)^2 = 0, \quad \mathcal{A}^{\alpha}(x) = \mathcal{A}(x)|_{U \to U^{e_\alpha}}$

- $4 \times \mathcal{A}^1(x)$
- $\mathcal{A}^2(x)$
- $\sum_\alpha e_\alpha \mathcal{A}^\alpha(x) = 4 \times \mathcal{A}^1(x) - \mathcal{A}^2(x)$



- Numerical computation of Weyl fermion measure

  Discrete interpolation of the admissible gauge fields

$$U_\mu^{t_i}(x) = \mathrm{e}^{it_i\tilde{A}_\mu(x)}V_{[m]}(x,\mu) \quad \{t_n,\cdots,t_i,\cdots,t_0|\ t_i\in[0,1]\}$$


  – Induced phase by the evolution of the chiral basis (projection operator)

$$\begin{aligned} \mathrm{e}^{i\phi} &= \det(v_k^1, Q_1 v_l^0) \\ &\fallingdotseq \frac{\det(v_k^1, \{\prod_i P_{t_i}\}v_l^0)}{|\det(v_k^1, \{\prod_i P_{t_i}\}v_l^0)|} \end{aligned}$$

  – $W$ : "Local counter term"

    The second term drops by the symmetric interpolation

$$U_\mu^{t,s}(x) = \mathrm{e}^{i\,st\,\tilde{A}_\mu(x)}V_{[m]}(x,\mu)$$

$$\begin{aligned} \ln W &= i\int_0^1 dt \sum_x \tilde{A}_\mu(x)\,\{k_\mu(x)\}\,|_{\tilde{A}\to t\tilde{A}} \\ &\qquad - \int_0^1\int_0^1 dsdt\,\mathrm{Tr}\left\{\hat{P}_-[\partial_t\hat{P}_-,\partial_s\hat{P}_-]\right\} \\ &\fallingdotseq i\sum_i \omega_i \sum_x \tilde{A}_\mu(x)\,\{k_\mu(x)\}\,|_{\tilde{A}\to t_i\tilde{A}} \end{aligned}$$

  – Chiral basis

$$v_i(x) = \begin{cases} v_1^1(x)\,\mathrm{e}^{i\phi}W^{-1} & \text{if } i=1 \\ v_i^1(x) & \text{otherwise} \end{cases}$$

## Gauge anomaly in non-abelian theories

- Non-abelian gauge anomaly (gauge-variant)

$$\sum_x \omega \cdot \nabla_\mu^* J_\mu = -i\mathrm{Tr}\omega\gamma_5 (1 - aD) + \sum_x \omega \cdot \nabla_\mu^* j_\mu$$

Non-abelian anomaly in 4 dim. $\Longrightarrow$ abelian anomaly in 6 dim.
Alvarez-Gaumé and Ginsparg

- Topological field on 4-dim. lattice plus two continuum dimensions : $\mathbb{L}^4 \times \mathbb{R}^2$

$$U_\mu(z), A_s(z), A_t(z) \qquad z = (x_\mu, s, t)$$

$$\sum_x q(z) = i\mathrm{Tr}\left\{\hat{P}_+ \left[\partial_t \hat{P}_+, \partial_s \hat{P}_+\right] - \frac{1}{2}\partial_t [A_s \hat{\gamma}_5] + \frac{1}{2}\partial_s [A_t \hat{\gamma}_5]\right\}$$

  - Cohomological analysis of $q(z)$ is required for exact gauge invariance
    - Generic non-abelian theories in all orders of lattice perturbation theory
      Suzuki, Lüscher
    - $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ electroweak theory (non-perturbative)
      Nakayama-Y.K.

  So far, any cohomological method applicable to non-abelian anomaly is not known

- **Admissibility-$\epsilon$ expansion**
  - Admissibility condition : $\epsilon$ is actually a small number!

$$|1 - P_{\mu\nu}(x)| \leq \epsilon, \qquad \epsilon \leq 1/30$$

  - Expansion in terms of the field tensor $F_{\mu\nu}(x) = \frac{1}{i}\ln P_{\mu\nu}(x)$

$$P_{\mu\nu}(x) = \mathrm{e}^{iF_{\mu\nu}(x)} = \sum_{l=0}^{\infty} \frac{1}{l!} i^l F_{\mu\nu}(x)^l$$

  - Link variables in the complete axial gauge (reference point $x_0$)

$$U_\mu(x) = \prod_{\mathrm{path}\{y\}} \mathrm{e}^{iF_{\mu\nu}^{(x_0)}(y)}, \qquad \mathrm{e}^{iF_{\mu\nu}^{(x_0)}(x)} = \Lambda(x, x_0)^{-1} \mathrm{e}^{iF_{\mu\nu}(x)} \Lambda(x, x_0)$$

  - A local expansion of $q(x)$

$$\begin{aligned} q(x) &= \theta^1_{\mu\nu}(x, y) \mathrm{Tr}\left(F^{(x)}_{\mu\nu}(y)\right) + \cdots \\ &\quad + \frac{1}{k!} \theta^k_{\mu_1\nu_1, \cdots, \mu_k\nu_k}(x, y_1, \cdots, y_k) \mathrm{Tr}\left(F^{(x)}_{\mu\nu}(y_1) \cdots F^{(x)}_{\mu\nu}(y_k)\right) + \cdots \end{aligned}$$

    * The coefficient bi-local fields are same as those in abelian case
    * Results of the cohomological analysis in abelian theory can be used

  - Chiral anomaly in two-dimensions : $q(x) = \mathrm{tr}\{\gamma_5(1 - aD)(x, x)\}$

$$\begin{aligned} \theta^1_{\mu\nu}(x, y) &= \partial^*_\lambda \tau_{\lambda\mu\nu}(x, y) + \gamma\epsilon_{\mu\nu}\delta_{x,y} \\ \theta^k_{\mu_1\nu_1, \cdots, \mu_k\nu_k}(x, y_1, \cdots, y_k) &= \partial^*_\lambda \tau^k_{\lambda\mu_1\nu_1, \cdots, \mu_k\nu_k}(x, y_1, \cdots, y_k) \quad (k \geq 2) \end{aligned}$$

Then we can show

$$q(x) = \partial^*_\lambda k_\lambda(x)$$

where

$$\begin{aligned} k_\lambda(x) &= \tau_{\lambda\mu\nu}(x, y) \mathrm{Tr}\left(F^{(x)}_{\mu\nu}(y)\right) + \cdots \\ &\quad + \frac{1}{k!} \tau^k_{\lambda\mu_1\nu_1, \cdots, \mu_k\nu_k}(x, y_1, \cdots, y_k) \mathrm{Tr}\left(F^{(x)}_{\mu\nu}(y_1) \cdots F^{(x)}_{\mu\nu}(y_k)\right) + \cdots \end{aligned}$$

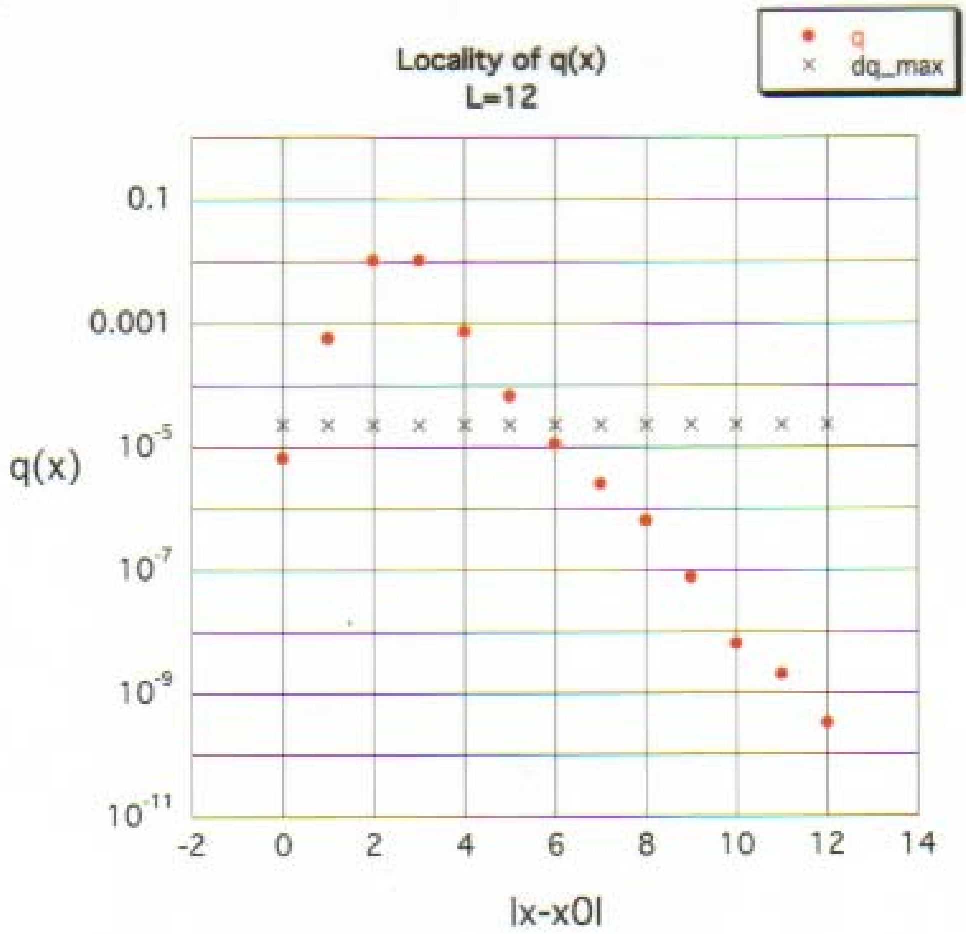
– Chiral anomaly in four-dimensions : $\boldsymbol{q(x) = \mathrm{tr}\,\{\gamma_5(1-aD)(x,x)\}}$

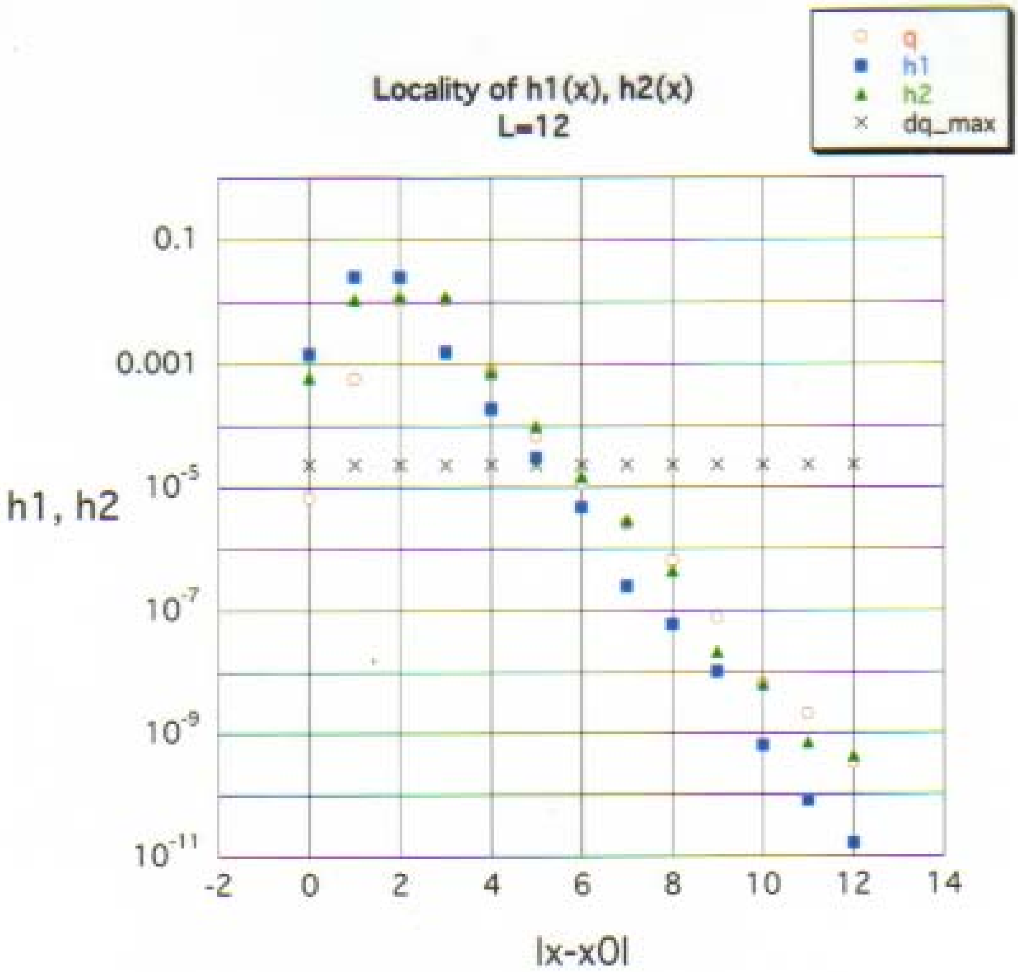
$$
\begin{aligned}
q(x) \;=\; & \frac{1}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\lambda\rho}(x+\hat{\mu}+\hat{\nu})\right) \\
& + c^2_{\sigma\mu\nu,\lambda\rho}(x)\mathrm{Tr}\left(\partial_\sigma F_{\mu\nu}(x)F_{\lambda\rho}(x)\right) \\
& + c^3_{\sigma\mu\nu,\lambda\rho,\tau\omega}(x)\mathrm{Tr}\left(\partial_\sigma F_{\mu\nu}(x)F_{\lambda\rho}(x)F_{\tau\omega}(x)\right) + \cdots \\
& + \partial^*_\mu k_\mu(x)
\end{aligned}
$$

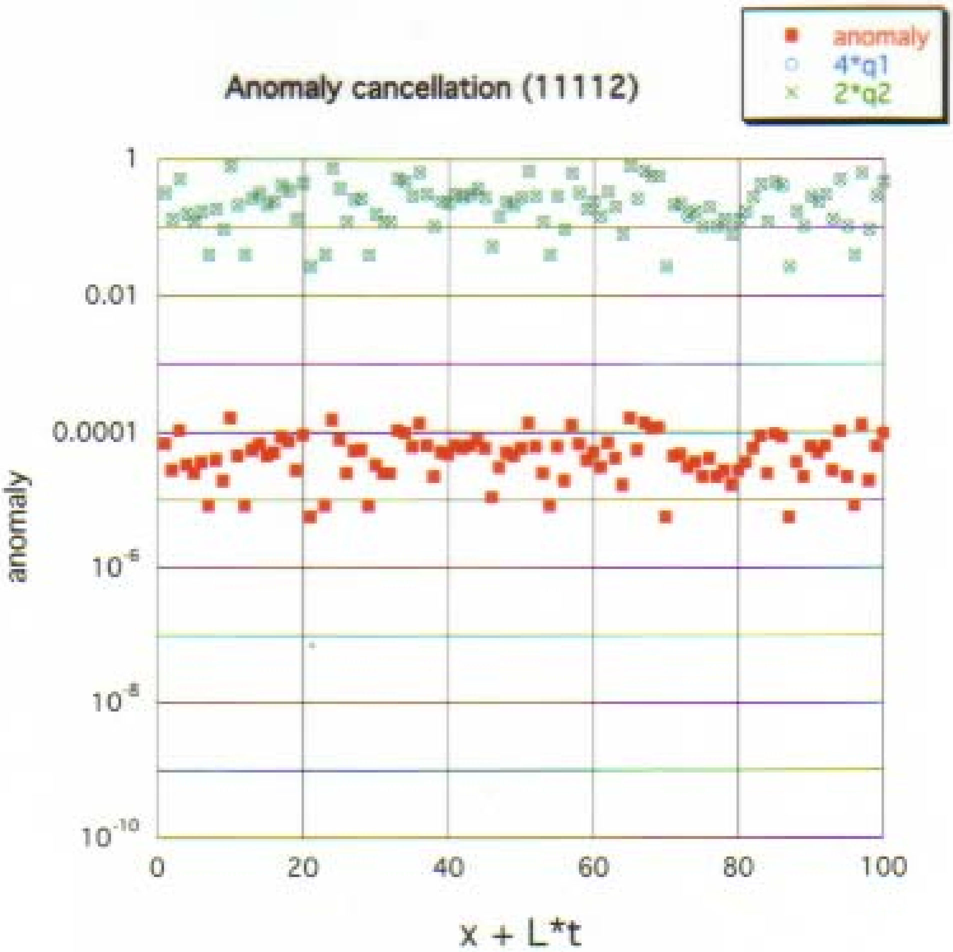
* Local, higher order terms remain
  (These terms vanish in U(1) case by the Bianchi identity )
* the Bianchi identity (?)

## Summary and Discussion

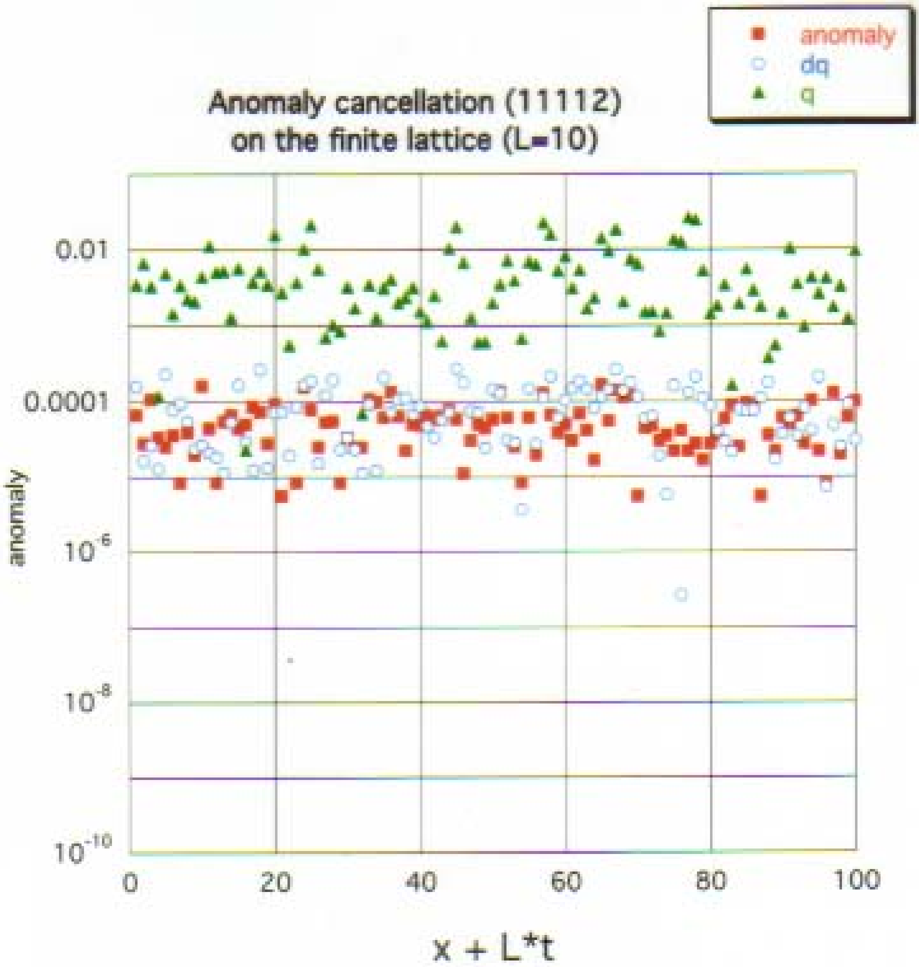
- Numerical implementation of $U(1)$ chiral gauge theories with exact gauge invariance
  - Cohomological analysis within the finite lattice
    * Poincaré lemma on the lattice
    * Vector potential representation of the link variables
  - A numerical solution to the local cohomology problem
    * Numerical computation of the local current $k_\mu(x)$ is possible
    * Locality and anomaly cancellation, checked in two-dimensions
  - Numerical computation of the gauge-invariant Weyl fermion measure
- Cohomological analysis of non-abelian gauge anomaly
  - Admissibility-$\epsilon$ expansion
    * two-dimensional chiral anomaly
    * four-dimensional chiral anomaly (?)
    *


Locality of q(x)
L=12
q
dq_max
0.1
0.001
$10^{-5}$
$10^{-7}$
$10^{-9}$
$10^{-11}$
q(x)
-2
0
2
4
6
8
10
12
14
|x-x0|


Locality of h1(x), h2(x)
L=12
q
h1
h2
dq_max
0.1
0.001
$10^{-5}$
$10^{-7}$
$10^{-9}$
$10^{-11}$
h1, h2
-2
0
2
4
6
8
10
12
14
|x-x0|



Anomaly cancellation (11112)
anomaly
4*q1
2*q2
1
0.01
0.0001
$10^{-6}$
$10^{-8}$
$10^{-10}$
anomaly
0
20
40
60
80
100
x + L*t



anomaly
dq
q
Anomaly cancellation (11112)
on the finite lattice (L=10)
0.01
0.0001
anomaly
10^-6
10^-8
10^-10
0
20
40
60
80
100
x + L*t