Pure Spinor Formalism and its Extension for Superstring

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1 Introduction

Gervais and Sakita (1971)

"Field Theory Interpretation of Supergauges in Dual Models"

Worldsheet super(conformal) symmetry in RNS string

$\Downarrow$

Wess and Zumino (1974)

Supersymmetry: generalized to 4 dimensions

$\Downarrow$

SUSY has become one of the central concepts in high energy physics

2-dimensions ⇒ 4-(and higher) dimensions

A drastic step in two respects:

1. SUSY representation

Going to higher dimensions ⇒ SUSY multiplet becomes large:

$$\Phi(x,\theta) = \phi(x) + \psi_\alpha(x)\theta^\alpha + \frac{1}{2}\rho_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \cdots$$

Two problems:

1. Fields with higher spins appear
2. SUSY-invariant kinetic term for low spin fields cannot be formed

   Example: $\phi =$ a scalar field

$$\phi\partial^2\phi = \int d^N\theta \Phi D^2 \Phi \ni \int d^N\theta \phi\partial^2\phi\theta^2$$

$\therefore \qquad N = 2 \Rightarrow$ 2-dimensions.

Above 2 dimensions, one must impose

SUSY-invariant constraints

to relate higher spin fields to derivatives of lower spin fields

♡ **Fortunately easy in 4D:**

Chiral (and anti-chiral) constraint: $\bar{D}_{\dot{\alpha}}\Phi^{(c)} = 0, \quad \Phi_1^{(c)}\Phi_2^{(c)}$ =chiral

♠ **Hard above 4D (or with R-symmetry):**

- Proper constraints are not easy to find
- Constrained superfields are hard to manipulate

◇ Only success: $4D\ \mathcal{N} = 2$ (or $6D\ \mathcal{N} = 1$) Harmonic superspace (Galperin et al)

– Enlarge the superspace $(x^\mu, \theta_i^\alpha, \bar{\theta}_i^\alpha)$ by adding a complex commuting $SU(2)$ spinor u_i^- with a constraint $u^{+i}u_i^- = 1, (u^{i+} = u_i^{-*})$.

– This allows one to define "analytic superfields" (= analogue of chiral superfields)

– Superfield has ∞ components but with appropriate measure for u_i^- one can construct proper action with finite number of fields.

Beyond 6D, no suceess so far $\Rightarrow$ difficulty for superstring (field) theory

2. String theory context

Worldsheet SUSY $\Rightarrow$ Target space SUSY

Apriori, these two notions are quite distinct.

$\Rightarrow$ **Two rather different (worldsheet) formulations of superstring**

Symmetries on the worldsheet and on the target space are connected in very subtle ways.

RNS

- Super-worldsheet, bosonic target manifold
  - Worldsheet SUSY is manifest: $\boldsymbol{X^\mu(z,\theta) = x^\mu(z) + \theta\psi^\mu(z)}$ and ghosts.
  - Spacetime SUSY emerges only after GSO projection, with infinite redundancy called "pictures".
  - Spinors are described by spin fields. RR (bispinor) fields are difficult to describe.
- Lorentz covariance is manifest:
- CFT fits well. BRST formalism is convenient and powerful

GS

- Bosonic worldsheet, target supermanifold
  - Spacetime SUSY is built in: $\boldsymbol{x^\mu, \theta^\alpha}$
  - To match up the worldsheet bosonic and fermionic DOF's, local fermionic $\boldsymbol{\kappa}$ symmetry is crucial.
- Classically Lorentz invariant. Constraints related to $\boldsymbol{\kappa}$-symmetry are inherently second class. Quantization difficult except in L.C. type gauge.
- CFT techniques are not applicable. Loop calculations are difficult.

So, after 30 years, supersymmetry in string theory is still a difficult subject of prime importance.

Other formulations should be possible

With D-brane physics becoming so important, one wishes to have a formulation where RR-fields can be treated with ease.

Berkovits: 2000 $\sim$:

Pure Spinor formalism

Claimed to be super-Poincaré covariant.

We will

- **Give a brief review of Berkovits' formalism**
- **Point out mysteries and unsolved problems of PS formalism**
- **Describe a new extension (EPS) toward more satisfactory formulation**

based on work with Yuri Aisaka

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2 Berkovits' Pure Spinor Formalism and Challenges

2.1 Two roots of Berkovits' formalism

□ **Siegel's Attempt (1986)**:

Classical GS action in conformal gauge:

$$\begin{aligned} S &= -\frac{T}{2}\left(\int d^2\xi\eta^{ij}\Pi_i^\mu\Pi_{\mu,j} - 2\left(\int \Pi^\mu\tilde{\Theta}_\mu + \frac{1}{2}\Theta^\mu\tilde{\Theta}_\mu\right)\right) \\ \Pi_i^\mu &= \partial_i x^\mu - i\bar{\theta}^a\Gamma^\mu\partial_i\theta^a \\ \Pi^\mu &= \Pi_i^\mu d\xi^i\,, \quad \Theta^\mu = i\bar{\theta}^a\Gamma^\mu d\theta^a\,, \quad \tilde{\Theta}^\mu = i\eta_a\bar{\theta}^a\Gamma^\mu d\theta^a\,, (\eta_1 = -\eta_2 = 1) \end{aligned}$$

$\Rightarrow$ Primary constraints: (restricting to holomorphic chiral sector),

$$d_\alpha \equiv p_\alpha + i\partial x_\mu(\gamma^\mu\theta)_\alpha + \frac{1}{2}(\gamma^\mu\theta)_\alpha(\theta\gamma_\mu\partial\theta) = 0 \quad (p_\alpha = \text{conj. to } \theta^\alpha)$$

Classically, $(\Pi^\mu = \partial x^\mu - i\theta\gamma^\mu\partial\theta)$

$$\{d_\alpha(z), d_\beta(w)\} = 2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)\delta(z-w)$$

♦ With Virasoro constraints $\Pi^\mu\Pi_\mu = 0$, $d_\alpha = 0$ are 8 first class and 8 second class, which cannot be separated covariantly.

♦ Action is not quadratic. Quantization possible on L.C. type gauge, totally fixed with the use of 8 κ=symmetries.

Siegel's proposal:

Superstring $\stackrel{?}{=}$ Free fields x^μ, $(\theta^\alpha, p_\alpha)$ with appropriate 1st class constraints

$$S = \frac{1}{\pi}\int d^2z \left(\frac{1}{2}\partial x^\mu \bar{\partial} x_\mu + p_\alpha \bar{\partial}\theta^\alpha\right)$$

$$T_{x,\theta} = -\frac{1}{2}(\partial x)^2 - p_\alpha \partial\theta^\alpha = -\Pi^\mu \Pi_\mu + \partial\theta^\alpha d_\alpha$$

Constraints: $\mathcal{A} = T_{x,\theta}\,, \mathcal{B}^\alpha = \Pi^\mu(\gamma_\mu d)^\alpha\,, \mathcal{C}_{\alpha\beta} = \frac{1}{2}(d_\alpha d_\beta - d_\beta d_\alpha)\,, \mathcal{D}^\mu = i(d\gamma^\mu \partial d)$:

Classically form "ABCD algebra".

Worked for superparticle but not for string quantum mechanically.
Some of its features, however, are utilized in PS formalism.

□ **Twistor String** (**1989**∼):

♦ 1989: Superparticle: Sorokin, Tkach, Volkov and Zheltukhin
Introduced a twistor-like variable= a bosonic spinor λ^α to express light-like momentum
$p^\mu = a\lambda\gamma^\mu\lambda \Rightarrow p^2 = 0$ $\Leftarrow$ Fierz identity
κ-symmetry $\Rightarrow$ worldsheet superconformal symmetry up to $N = 8$

♦ 1989 ∼ 1992: Berkovits; Tonin; Delduc et al: $N = 2$ version for superstring

♦ 1992 ∼ 1999: Mainly Berkovits:
"Hybrid formalism" : $N = 2$ Lorentz covariant twistor string for 4D ⊕ RNS for 6D.

♡ **Finally, in 2000, Berkovits gathered hints from these two ideas to invent a new formalism using "pure spinor".**

- Concept of pure spinor dates back to Cartan's famous treatise.
- Howe recognized (1991) the usefulness of pure spinor for describing on-shell conditions for SYM and SG.

2.2 What is a pure spinor ?

E. Cartan: *Leçon sur la Theorie des Spineurs* , 1937.

On the way to construct spinor representations, Cartan introduced pure spinors as quantities characterizing **null vectors and planes** in complexified Euclidean space.

□ **Maximal null-plane in $E_{2\nu+1}$** :

In $E_{2\nu+1}$, maximum number of mutually orthogonal null vectors is ν and they span the maximal null-plane N_ν.

Take "light-cone" coordinates $(u_i = x_i + ix_{\nu+i}, v_i = x_i - ix_{\nu+i})$ for 2ν-dimensional part:

$$F \equiv x^2 = (x^0)^2 + \sum_{i=1}^{\nu} u^i v^i$$

◇ $\nu + 1$ conditions $x^0 = v^i = 0 \Rightarrow$ simplest null plane N_ν.

More general way to get a N_ν: If (x^0, u^i) are arbitrary, no way to get a null plane $\Rightarrow$ Need at least one linear relation among x^0, u^i. Write it as

$$\boxed{(\star 1) \quad \eta_0 \equiv \lambda_0 x^0 + \sum_i \lambda_i u^i = 0}$$

Then, $\lambda_0 F = (\lambda_0 x^0) x^0 + \sum_i u^i (\lambda_0 v^i) = \sum_i u^i \chi_i$ with $\chi_i = \lambda_0 v^i - \lambda_i x^0$.
For this to vanish, we need $\chi_i = -\sum_j \lambda_{ik} u^k$, or

$$\boxed{(\star 2) \quad \eta_i \equiv \chi_i + \sum_k \lambda_{ik} u^k = 0\,, \qquad \text{where} \qquad \lambda_{ik} = -\lambda_{ki}}$$

The $\nu+1$ relations $(\star 1)$ and $(\star 2)$ define a null ν-plane in terms of $\lambda_0, \lambda_i, \lambda_{ij}$.
There are $1 + \frac{1}{2}\nu(\nu+1)$ components. (For 10D chiral, $\nu = 4 \Rightarrow 11$ components.)

♡ Cartan then gave rules to construct 2^ν dimensional spinor out of λ_0, λ_i and λ_{ij}.

◇ **Case of $E_{2\nu+2}$:** $\Rightarrow$ chiral and anti-chiral null planes. Each has structure similar to the one in $E_{2\nu+1}$.

◇ Conversely, Cartan showed

Characterization of Pure Spinor

a spinor λ is pure $\Leftrightarrow$ $\mathcal{F}_p \equiv \lambda^T C\Gamma^{(p)}\lambda$ vanishes for $p < \nu$

$\mathcal{F}_p$ sometimes vanishes identically

- due to anti-symmetry of $C\Gamma^{(p)}$
- when λ is restricted to be chiral (or anti-chiral).

Pure spinors in 10 and 11 dimensions

10-dimensions	For a chiral spinor,	$\lambda^T C\Gamma^\mu\lambda = 0$
11-dimensions	$\lambda^T C\Gamma^\mu\lambda = 0\,,$	$\lambda^T C\Gamma^{\mu\nu}\lambda = 0$

Pure spinors are reminiscent of harmonic variables

2.3 Basic ingredients of PS formalism

□ **BRST-like charge Q**:

Berkovits' proposal:

Physical states of superstring = cohomology of Q

$$Q = \int [dz] j(z), \qquad j(z) = \lambda^\alpha(z) d_\alpha(z)$$

λ^α =16 component chiral bosonic pure spinor:

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0, \qquad \Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^{\alpha\beta} \\ (\gamma^\mu)_{\alpha\beta} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda^\alpha \\ \lambda_\alpha \end{pmatrix}, \quad \alpha,\beta = 1 \sim 16$$

Spinor covariant derivative:

$$d_\alpha = p_\alpha + i\partial x_\mu (\gamma^\mu \theta)_\alpha + \frac{1}{2} (\gamma^\mu \theta)_\alpha (\theta \gamma_\mu \partial \theta)$$

Spacetime SUSY:

$$\delta\theta^\alpha = \epsilon^\alpha, \qquad \delta x^\mu = i\epsilon\gamma^\mu\theta$$

Free fields:

$$x^\mu(z) x^\nu(w) = -\eta^{\mu\nu} \ln(z-w), \qquad p_\alpha(z) \theta^\beta(w) = \frac{\delta_\alpha^\beta}{z-w}$$

OPE of d_α with itself:

$$d_\alpha(z)d_\beta(w) = \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w}$$

Superinvariant "momentum": ⋆

$$\Pi_\mu = \partial x_\mu - i\theta\gamma_\mu\partial\theta$$

OPE of basic superinvariants of dimension 1:

$$d_\alpha(z)d_\beta(w) = \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w}$$

$$d_\alpha(z)\Pi^\mu(w) = \frac{-2i(\gamma^\mu\partial\theta)_\alpha(w)}{z-w}$$

$$\Pi^\mu(z)\Pi^\nu(w) = -\frac{\eta^{\mu\nu}}{(z-w)^2}$$

$$d_\alpha(z)\partial\theta^\beta(w) = \frac{\delta^\beta_\alpha}{(z-w)^2}$$

□ **Q is nilpotent with PS constraints**:

$$\begin{aligned} Q^2 &= \int [dz] \int [dw] \lambda^\alpha(z)\lambda^\beta(w) d_\alpha(z) d_\beta(w) \\ &= 2i \int [dw] \lambda^\alpha(w)\gamma^\mu_{\alpha\beta}\lambda^\beta(w)\Pi_\mu(w) = 0 \end{aligned}$$

$\Rightarrow$ One can consider constrained cohomology of Q.

Using $SO(8)$ parametrization of PS, Berkovits showed

Constrained cohomology of Q $=$ light-cone states of superstring

(Proof required infinite number of ghosts.)

2.4 Vertex operators and scattering amplitudes

Construction of the vertex operators owes much to Siegel's old attempt at covariant quantization.

□ **Unintegrated vertex operator for massless modes** :

Vertex op. for super-Maxwell multiplet:

$$U = \lambda^\alpha A_\alpha(x, \theta)$$

$A_\alpha =$ a spinor superfield satisfying the "on-shell" condition

$$A_\alpha = \psi_\alpha + B_{\alpha\beta}\theta^\beta + W_{\alpha\beta\gamma}\theta^\beta\theta^\gamma + \cdots$$

on-shell cond. $\qquad (\gamma^{\mu_1\mu_2\cdots\mu_5})^{\alpha\beta} D_\alpha A_\beta = 0$

where

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\gamma^\mu\theta)_\alpha \frac{\partial}{\partial x^\mu} \, . \\ \{D_\alpha, D_\beta\} &= -2i\gamma^\mu_{\alpha\beta}\partial_\mu \end{aligned}$$

$\boldsymbol{U}$ is $\boldsymbol{Q}$-invariant with PS and on-shell conditions:

$$
\begin{aligned}
QU(w) &= \int [dz]\lambda^{\alpha}(z)d_{\alpha}(z)\lambda^{\beta}(w)A_{\beta}(w) \\
&= \lambda^{\alpha}\lambda^{\beta}\underbrace{\frac{1}{2}(D_{\alpha}A_{\beta}+D_{\beta}A_{\alpha})}_{C_{\alpha\beta}}
\end{aligned}
$$

General expansion of a symmetric bispinor $C_{\alpha\beta}$ in 10D:

$$
\begin{aligned}
C_{\alpha\beta} &= C_{\mu}(\gamma^{\mu})_{\alpha\beta}+\frac{1}{5!}C_{\mu_5\mu_4\ldots\mu_1}(\gamma^{\mu_1\mu_2\ldots\mu_5})_{\alpha\beta}\,, \\
C_{\mu} &= \frac{1}{16}(\gamma_{\mu})^{\alpha\beta}C_{\beta\alpha}\,, \quad C_{\mu_5\ldots\mu_1} = \frac{1}{32}(\gamma_{\mu_5\ldots\mu_1})^{\alpha\beta}C_{\beta\alpha}\,.
\end{aligned}
$$

$$
\begin{aligned}
QU &= \frac{1}{2}\lambda^{\alpha}\lambda^{\beta}(D_{\alpha}A_{\beta}+D_{\beta}A_{\alpha}) \\
&= \frac{1}{16}(\lambda\gamma^{\mu}\lambda)(\gamma^{\mu})^{\alpha\beta}D_{\alpha}A_{\beta} \\
&\quad +\frac{1}{32\times 5!}(\lambda\gamma^{\mu_1\mu_2\ldots\mu_5}\lambda)(\gamma_{\mu_5\ldots\mu_1})^{\alpha\beta}D_{\alpha}A_{\beta} = 0
\end{aligned}
$$

Consequences of on-shell condition $(\gamma^{\mu_1\mu_2\cdots\mu_5})^{\alpha\beta}D_\alpha A_\beta = 0$:

$$
\begin{aligned}
&(i) \quad D_\alpha A_\beta + D_\beta A_\alpha + 2i\gamma^\mu_{\alpha\beta}B_\mu = 0 \\
&(ii) \quad D_\alpha B_\mu - \partial_\mu A_\alpha + 2i(\gamma_\mu)_{\alpha\beta}W^\beta = 0\,. \\
&(iii) \quad D_\alpha W^\beta - \frac{1}{4}(\gamma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu} = 0 \\
&(iv) \quad (\gamma^\mu)_{\gamma\alpha}\partial_\mu W^\beta = \frac{i}{8}\left((\gamma^{\mu\nu})_\alpha{}^\beta D_\gamma F_{\mu\nu} + (\gamma^{\mu\nu})_\gamma{}^\beta D_\alpha F_{\mu\nu}\right) \\
&(v) \quad (\gamma^\rho)_{\alpha\gamma}(\gamma^{\mu\nu})_\delta{}^\beta\partial_\rho F_{\mu\nu} = \left((\gamma^{\mu\nu})_\alpha{}^\beta(\gamma^\rho)_{\delta\gamma} + (\alpha\leftrightarrow\gamma)\right)\partial_\rho F_{\mu\nu} \\
&\qquad\qquad + \frac{1}{2}\left((\gamma^{\sigma\rho})_\alpha{}^\beta(\gamma_\sigma)_{\delta\kappa}(\gamma^{\mu\nu})_\gamma{}^\kappa + (\alpha\leftrightarrow\gamma)\right)\partial_\rho F_{\mu\nu}
\end{aligned}
$$

$B_\mu = (i/16)\gamma_\mu^{\alpha\beta}D_\alpha A_\beta =$ gauge superfield

$W^\alpha = (i/20)(\gamma^\mu)^{\alpha\beta}(D_\beta B_\mu - \partial_\mu A_\beta) =$ gaugino superfield

$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu =$ field strength superfield

Equations of motion

$(iv) \quad \Rightarrow \quad (\gamma^\mu)_{\alpha\beta}\partial_\mu W^\beta = 0$

$(v) \quad \Rightarrow \quad \partial^\mu F_{\mu\nu} = 0$

Gauge transformation:

$$\delta U = \delta(\lambda^\alpha A_\alpha) = Q\Lambda = \lambda^\alpha D_\alpha \Lambda$$

$$\therefore \qquad \delta A_\alpha = D_\alpha \Lambda$$

This contains the usual gauge transformation for B_μ:

$$\begin{aligned}\delta B_\mu &= \frac{i}{16}\gamma_\mu^{\alpha\beta} D_\alpha \delta A_\beta = \frac{i}{16}\gamma_\mu^{\alpha\beta} D_\alpha D_\beta \Lambda \\ &= \frac{1}{16}\mathrm{Tr}\,(\gamma^\mu \gamma_\mu)\partial_\mu \Lambda = 10\partial_\mu \Lambda\end{aligned}$$

□ **Integrated vertex operator $\int [dz] V(z)$:**

Needed for calculation of n-point amplitudes for $n \geq 4$

$V(z)$ is defined as

$$QV = \partial U = \partial(\lambda^\alpha A_\alpha) \Rightarrow Q\int [dz] V = 0$$

Solution (utilizing $(i) \sim (iv)$):

$$V = \underbrace{\partial\theta^\alpha A_\alpha + \Pi^\mu B_\mu + d_\alpha W^\alpha}_{\text{Siegel}} + \frac{1}{2} L^{\mu\nu}_{(\lambda)} F_{\mu\nu}$$

$L^{\mu\nu}_{(\lambda)}$ = Lorentz generator for pure spinor sector

□ **Covariant rules for calculation of scattering amplitudes** :

Scattering amplitude:

$$\mathcal{A} = \langle U_1(z_1) U_2(z_2) U_3(z_3) \int [dz_4] V_4(z_4) \cdots \int [dz_N] V_N(z_N) \rangle \,,$$

Integrate over the non-zero modes

$$\mathcal{A} = \int dz_4 \cdots \int dz_N \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta) \rangle$$

Prescription over the zero modes

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta) \rangle \;=\; \equiv \int d\Omega_\lambda (\bar{\lambda}_\delta \lambda^\delta)^{-3} \left(\frac{\partial}{\partial\theta} \gamma^{\mu\nu\rho} \frac{\partial}{\partial\theta} \right) \left(\bar{\lambda} \gamma_\mu \frac{\partial}{\partial\theta} \right)$$
$$\times \left(\bar{\lambda} \gamma_\nu \frac{\partial}{\partial\theta} \right) \left(\bar{\lambda} \gamma_\rho \frac{\partial}{\partial\theta} \right) \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)$$

Can show that this prescription is supersymmetric and reproduces known amplitudes

☐ Other results and applications: :

- Vertex operator for the first massive level
- Relation to RNS (not complete, involving singular functions)
- Easy to apply to a superparticle
- Derivation of 10D supergravity constraints
- Derivation of SUSY Born-Infeld action
- Perturbatively quantizable action in $\mathbf{AdS_5} \times \boldsymbol{S^5}$ background, with infinite number of conserved non-local charges.
- Application to supermembrane has been proposed. Quantization is difficult
  $\Rightarrow$ Matrix regularization ? (Appears non-trivial)

2.5 Challenges for PS Formalism

Many intricate and remarkable successes already obtained makes one feel that the PS formalism is a promising candidate for covariant description of superstring.
However, there are many basic questions to be answered.

The biggest question is

Where does it come from ?

- Underlying symmetry ?
- Underlying action ?
- Where is the Virasoro algebra ?
- How to quantize (with PS constraints) ?
- What is Q a “BRST” charge of ?

Many other questions stem from this lack of understanding of the fundamental origin of the formalism.

- All the fields are free quantized fields. How does one actually treat the PS constraints in the quantization procedure ?
- Is it really super-Poincaré covariant ? Intermediate steps are only $U(5)$-covariant. $U(5)$ decomposition of a chiral $SO(10)$ spinor:

$$|\lambda\rangle = \underbrace{\lambda_+|+\rangle}_{1} + \frac{1}{2}\underbrace{\lambda_{IJ}|IJ\rangle}_{10} + \underbrace{\lambda_{\tilde{I}}|\tilde{I}\rangle}_{\bar{5}}, \qquad (I, J, \tilde{I} = 1 \sim 5)$$

PS $\Leftrightarrow$ $\lambda_{\tilde{I}} = \frac{1}{8}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\lambda_{LM} =$ composite fields

Only the "rules" are stated covariantly. How does one derive these rules ?

- How to compute loop amplitudes ?
- What is the Hilbert space structure ? What is the proper inner product ?
- How to describe D-branes in this formalism ?

A number of problems above can be solved
if PS constraints are removed.
Can we do that consistently ?

3 An Extension of Pure Spinor Formalism without PS constraints

- Non-trivial to remove PS constraints:

Without PS constraints, $j(z)j(w) = 2i(\lambda\gamma^{\mu}\lambda)(w)\Pi_{\mu}(w)/(z-w) \neq 0$.
$\Rightarrow$ The algebra of $d_{\alpha}, \Pi^{\mu}, \partial\theta^{\alpha}$ contains second class pieces (*i.e.* central terms). ⋆
Cannot apply BV procedure to construct a nilpotent charge with non-trivial cohomology:

Our basic idea: Turn it into an appropriate first class algebra of constraints.

3.1 $U(5)$ analysis of PS constraints

PS constraints: $\Lambda^{\mu} \equiv \lambda\gamma^{\mu}\lambda = 0$

There should only be $5 (= 16 - 11)$ independent constraints. In fact one can write

Independent PS constraints

$$\Lambda^{\mu} = \mathcal{N}_I^{\mu}\Phi_I$$
$$\Phi_I \equiv \lambda_+\lambda_{\tilde{I}} - \frac{1}{8}\epsilon_{IJKLM}\lambda_{JK}\lambda_{LM}$$

$$\mathcal{N}_I^{\mu} \equiv -4(e_I^{+\mu} - \lambda_+^{-1}\lambda_{IJ}e_J^{-\mu})$$
$$\bar{\mathcal{N}}_I^{\mu} \equiv -\frac{1}{2}e_I^{-\mu}$$
$$e_I^{\pm\mu} \equiv \frac{1}{2}(\delta_{\mu,2I-1} \pm i\delta_{\mu,2I})$$

connect $SO(10)$ vector to $5 \oplus \bar{5}$ of $U(5)$

$\mathcal{N}_I^{\mu}$ and $\bar{\mathcal{N}}_I^{\mu}$ are null vectors satisfying orthonormality and completeness:

$$\mathcal{N}_I^{\mu}\mathcal{N}_J^{\mu} = 0\,, \quad \bar{\mathcal{N}}_I^{\mu}\bar{\mathcal{N}}_J^{\mu} = 0\,,$$
$$\mathcal{N}_I^{\mu}\bar{\mathcal{N}}_J^{\mu} = \delta_{IJ}\,,$$
$$\mathcal{N}_I^{\mu}\bar{\mathcal{N}}_I^{\nu} + \mathcal{N}_I^{\nu}\bar{\mathcal{N}}_I^{\mu} = \delta^{\mu\nu}$$

3.2 New first class algebra

Recall the second class algebra formed by the basic superinvariants

$$
\begin{aligned}
d_\alpha(z)d_\beta(w) &= \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w} \\
d_\alpha(z)\Pi^\mu(w) &= \frac{-2i(\gamma^\mu\partial\theta)_\alpha(w)}{z-w} \\
\Pi^\mu(z)\Pi^\nu(w) &= -\frac{\eta^{\mu\nu}}{(z-w)^2} \\
d_\alpha(z)\partial\theta^\beta(w) &= \frac{\delta_\alpha^\beta}{(z-w)^2}
\end{aligned}
$$

We construct a **new first class algebra** with λ^α **without PS constratins**

Define $j(z) = \lambda^\alpha(x)d_\alpha(z)$. Then

$$
j(z)j(w) = \frac{2i\Lambda^\mu\Pi_\mu(w)}{z-w}, \qquad (\Lambda^\mu = \lambda\gamma^\mu\lambda)
$$

Using the decomposition $\Lambda^\mu = \mathcal{N}_I^\mu\Phi_I$, introduce a new operator

$$
\mathcal{P}_I \equiv \mathcal{N}_I^\mu\Pi_\mu
$$

Then

$$j(z)j(w) = \frac{2i\mathcal{P}_I\Phi_I(w)}{z-w}$$

Now

$$\mathcal{P}_I(z)\mathcal{P}_J(w) = -\frac{\mathcal{N}_I^\mu(z)\mathcal{N}_J^\mu(w)}{(z-w)^2} = \frac{\mathcal{S}_{IJ}(w)}{z-w} \qquad \Leftarrow \text{null nature of } \mathcal{N}_I^\mu$$

$$\mathcal{S}_{IJ} \equiv -(\partial\mathcal{N}_I^\mu)\mathcal{N}_J^\mu$$

Further, using a non-trivial identity we get

$$j(z)\mathcal{P}_I(w) = \lambda^\alpha\mathcal{N}_I^\mu d_\alpha(z)\Pi_\mu(w) = \frac{-2i\mathcal{N}_I^\mu(\lambda\gamma_\mu\partial\theta)}{z-w}$$

$$= \frac{\mathcal{R}_{IJ}\Phi_J(w)}{z-w}, \qquad \mathcal{R}_{IJ} \equiv 2i\lambda_+^{-1}\mathcal{N}_I^\mu(\gamma_\mu\partial\theta)_J$$

In this way we get

New closed first class algebra

$$
\begin{aligned}
j(z)j(w) &= \frac{2i\mathcal{P}_I\Phi_I}{z-w} \\
j(z)\mathcal{P}_I(w) &= \frac{\mathcal{R}_{IJ}\Phi_J}{z-w} \\
j(z)\mathcal{R}_{IJ}(w) &= \frac{-i\mathcal{S}_{IJ}}{z-w} \\
\mathcal{P}_I(z)\mathcal{P}_J(w) &= \frac{\mathcal{S}_{IJ}}{z-w} \\
\text{all the rest} &= \text{non-singular}
\end{aligned}
$$

3.3 Nilpotent BRST-like Charge and its Cohomology

- Batalіln-Vilkovisky procedure:

Ghost-antighost pair for each generator $j, \mathcal{P}_I, \mathcal{R}_{IJ}, \mathcal{S}_{IJ} \Rightarrow Q' : \quad Q'^2 = 0$

□ **Better method with less ghosts: Homological perturbation** :

General set up: Suppose $^\exists$ 3 fermionic operators (δ, Q, d_1) of degrees $(-1, 0, 1)$, satisfying the following properties: ($AB \equiv$ OPE of A and $B = [A, B\}$,
Graded Jacobi: $ABC \equiv A(BC) = (AB)C \pm B(AC)$)

$$(i) \qquad \delta^2 = 0$$

$$(ii) \qquad \delta A_n = 0 \quad \Rightarrow \quad A_n = \delta B_{n+1} \quad \text{for } n \geq 1$$

$\Leftrightarrow$ **δ-homology** can be defined and is trivial for $n \geq 1$.

$$(iii) \qquad \delta Q = 0 \qquad (Q \text{ and } \delta \text{ anti-commute})$$

$$(iv) \qquad Q^2 = -2\delta d_1 \qquad (Q^2 \text{ is non-vanishing but } \delta\text{-exact})$$

Then,

♡ **Main theorems of homological perturbation theory:**

♦ Nilpotent $\hat{Q}$ can be constructed by "perturbation" in degree as

$$\hat{Q} = \delta + Q + d_1 + d_2 + d_3 + \cdots$$

♦ **Cohomology of $\hat{Q}$ = Cohomology of Q with $\delta = 0$**

Application to our system: Introduce 5 anti-commuting pairs (c_I, b_I) and asign $\deg c_I = 1\,, \deg b_I = -1$. The following operators satisfy all the assumptions:

$$\begin{aligned}\delta &= \int [dz](-i\Phi_I b_I) \\ Q &= \int [dz] j \\ d_1 &= \int [dz] c_I \mathcal{P}_I\end{aligned}$$

Sketch of how perturbation works:

Want to achieve

$$0 \;=\; \hat{Q}^2 = \underbrace{Q^2 + 2\delta d_1}_{\text{degree } 0} + \underbrace{2(\delta d_2 + Q d_1)}_{\text{degree } 1} + \cdots$$

- Degree 0: Satisfied due to $(*)$.
- Degree 1: Act δ on $Qd_1 \Rightarrow \delta(Qd_1) = -Q(\delta d_1) = \frac{1}{2}Q^3 = 0$

Triviality of δ homology $\Rightarrow Qd_1 = -\delta X_2$

$\Rightarrow 0 = \delta(d_2 - X_2) \Rightarrow d_2 = X_2 + \delta Y_3$ (will set $Y_3 = 0$)

In our case, explicitly,

$$Qd_1 \;=\; -\int [dz] c_I R_{IJ} \Phi_J = -\delta X_2$$

$$d_2 \;=\; X_2 = -\frac{i}{2}\int [dz] c_I c_J R_{IJ}$$

Perturbation terminates at one step !

Nilpotent $\hat{Q}$

$$\begin{aligned}\hat{Q} &= \delta + Q + d_1 + d_2 \\ &= \int [dz] \left(-ib_I \Phi_I + j + c_I \mathcal{P}_I - \frac{i}{2} c_I c_J R_{IJ} \right)\end{aligned}$$

Constraints $\Phi_I = 0$ in λ-space are removed with a minimum number of ghosts (c_I, b_I)
Cohomology of $\hat{Q}$ is guaranteed to be the same as Berkovits' constrained cohomology

3.4 Action and energy-momentum tensor

Euclidean action:

$$S = \frac{1}{\pi} \int d^2 z \left(\frac{1}{2} \partial x_\mu \bar{\partial} x^\mu + p_\alpha \bar{\partial} \theta^\alpha + \omega_\alpha \bar{\partial} \lambda^\alpha + b_I \bar{\partial} c_I \right)$$

EM tensor:

$$T = -\frac{1}{2}\partial x^\mu \partial x_\mu - p_\alpha \partial\theta^\alpha - \omega_\alpha \partial\lambda^\alpha - b^\mu \partial c_\mu$$
$$= -\frac{1}{2}\Pi^\mu \Pi_\mu - d_\alpha \partial\theta^\alpha - \omega_\alpha \partial\lambda^\alpha - b_I \partial c_I$$

The system forms a free centerless CFT.

Only the (b_I, c_I) ghost sector breaks manifest Lorentz invariance.

3.5 Vertex Operators for Massless Modes

□ **Unintegrated vertex operator**:

Recall massless unintegrated vertex operator of Berkovits

$$U_0 = \lambda^\alpha A_\alpha(x, \theta)$$

Need to extend this to $U = \sum_{n=0}^{5} U_n$ satisfying

$$0 = \hat{Q}U \Rightarrow 0 = QU_n + d_1 U_{n-1} + d_2 U_{n-2} + \delta U_{n+1}$$

Construction again utilizes δ-homology logic:

- From nilpotency of $\hat{Q}$ one can check $\delta(QU_n + d_1U_{n-1} + d_2U_{n-2}) = 0$. $\Rightarrow U_{n+1}$ exists.
- Actually compute $QU_n + d_1U_{n-1} + d_2U_{n-2}$ and identify U_{n+1}.

Explicit form of U

$$\begin{aligned}
U_0 &= \lambda^\alpha A_\alpha \qquad\qquad (c^\mu \equiv \mathcal{N}_I^\mu c_I) \\
U_1 &= -c^\mu B_\mu \\
U_2 &= -\lambda_+^{-1} c^\mu c^\nu \bar{\mathcal{N}}_I^\nu (\gamma_\mu W)_I \\
U_3 &= \frac{i}{12}\lambda_+^{-2} c^\mu c^\sigma c^\tau \bar{\mathcal{N}}_I^\sigma \bar{\mathcal{N}}_J^\tau (\gamma_{\mu\nu\rho})_{IJ} F^{\nu\rho} \\
U_4 &= \frac{i}{12}\lambda_+^{-3} c^{\mu_1} c^{\mu_2} c^{\mu_3} c^\mu \bar{\mathcal{N}}_{I_1}^{\mu_1} \bar{\mathcal{N}}_{I_2}^{\mu_2} \bar{\mathcal{N}}_{I_3}^{\mu_3} (\gamma_{\mu\nu\rho})_{I_1 I_2} (\gamma^\nu \partial^\rho W)_{I_3}
\end{aligned}$$

□ Integrated vertex operator, "b-ghost" and Virasoro operator :

Integrated vertex operator $\int [dz] V(z)$

Characterized by

$$(\heartsuit) \qquad \hat{Q} V(z) = \partial U(z)\,, \qquad U(z) = \text{unintegrated vertex}$$

$V(z)$ can be easily constructed if we find the "b-ghost" field $B(z)$ with the property

$$T(z) = \hat{Q} B(z) \equiv \left\{ \hat{Q}, B(z) \right\}$$

Define $\mathcal{B} \equiv \int [dz] B(z)$. Then, $\hat{Q}\mathcal{B} = \int [dz] T(z) = L_{-1}$ which acts as ∂. Therefore

$$\partial U = L_{-1} U = (\hat{Q}\mathcal{B}) U = \hat{Q}(\mathcal{B}U) + \mathcal{B}(\hat{Q}U) = \hat{Q}(\mathcal{B}U)$$

$$\therefore \qquad V = \mathcal{B} U \quad (+\hat{Q}\Omega)$$

Such $B(z)$ indeed exists in our extended formalism in remarkably simple form:

$B(z)$ ghost

$$B(z) = -\omega_\alpha \partial \theta^\alpha - b_I \bar{\mathcal{N}}_I^\mu \Pi_\mu$$

$\omega_\alpha =$ conjugate to λ^α

- Such $B(z)$ cannot be constructed in Berkovits' formalism.
- Relation between "BRST operator" and T is clarified for the first time in PS formalism.
- We expect "b-ghost" to play crucial roles, as in RNS formalism.

Explicit form of V

$$\begin{aligned} V &= \mathcal{B}U + \hat{Q}(\omega W + b^\mu B_\mu) \\ &= V_0^{\text{Berkovits}} - b^\mu c^\nu F_{\mu\nu} + \frac{1}{2}\partial\widetilde{\mathcal{N}}^{\mu\nu} F_{\mu\nu} + \text{(degree higher than 1)} \end{aligned}$$

☐ Twisted $N = 2$ SCA Structure:

$$
\begin{aligned}
T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
T(z)G(w) &= \frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \quad T(z)\bar{G}(w) = \frac{2\bar{G}(w)}{(z-w)^2} + \frac{\partial \bar{G}(w)}{z-w} \\
T(z)J(w) &= \frac{-d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \\
G(z)\bar{G}(w) &= \frac{2d}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)}{z-w} \\
J(z)G(w) &= \frac{G(w)}{z-w}, \quad J(z)\bar{G}(w) = \frac{-\bar{G}(w)}{z-w}, \quad J(z)J(w) = \frac{d}{(z-w)^2}
\end{aligned}
$$

This is realized in EPS by

$$
\begin{aligned}
G &= \hat{q} = \text{current of } \hat{Q}\,, \qquad \bar{G} = 2B \\
J &= -\lambda^\alpha \omega_\alpha - b_I c_I = \text{ghost number current} \\
d &= -11
\end{aligned}
$$

This structure might prove useful in future development.

3.6 Relation to RNS Formalism

Wish to establish a precise correspondence with RNS.

- ♦ Another proof of the correct physical spectrum.
- ♦ Hints for understanding the zero-mode rules for amplitude computation.

♠ Relation is bound to be quite non-trivial

- vector $\leftrightarrow$ spinor
- $\#$ of DOF are different

♠ Berkovits' previous attempt (JHEP 0108(01) 026) involved singular operations, due to PS constraints.

$\Downarrow$

Expect better situation for EPS

□ **Field content**:

$$\text{EPS} \quad \underset{(10)}{x_I^{\pm}} \quad \underset{(-10)}{\begin{pmatrix} \theta_{\tilde{I}} \\ p_I \end{pmatrix}} \quad \underset{(-2)}{\begin{pmatrix} \theta_+ \\ p_- \end{pmatrix}} \quad \underset{(2)}{\begin{pmatrix} \lambda_+ \\ \omega_- \end{pmatrix}} \quad \underset{(-20)}{\begin{pmatrix} \theta_{IJ} \\ p_{\tilde{I}\tilde{J}} \end{pmatrix}} \quad \underset{(20)}{\begin{pmatrix} \lambda_{IJ} \\ \omega_{\tilde{I}\tilde{J}} \end{pmatrix}} \quad \underset{(10)}{\begin{pmatrix} \lambda_{\tilde{I}} \\ \omega_I \end{pmatrix}} \quad \underset{(-10)}{\begin{pmatrix} c_I \\ b_I \end{pmatrix}}$$

$$\text{RNS} \quad \underset{(10)}{x_I^{\pm}} \quad \underset{(5)}{\psi_I^{\pm}} \quad \underset{(-26)}{\begin{pmatrix} c \\ b \end{pmatrix}} \quad \underset{(11)}{\begin{pmatrix} \beta \\ \gamma \end{pmatrix}}$$

Can we connect them by a similarity transformation ?

□ **Simplifying BRST operator on EPS side**:

Need to decouple the two quartets in order to connect to RNS.

Key ideas:

- Classify terms in $\hat{Q}$ by new judicious degrees.
- Decompose the nilpotency relation $\hat{Q}^2 = 0$ according to the new degree.
- Make systematic use of δ-homology idea to reduce the cohomology.

1. Decoupling of $\{p_{\tilde{I}\tilde{J}}, \theta_{IJ}, \lambda_{IJ}, \omega_{\tilde{I}\tilde{J}}\}$ = quartet w.r.t $\tilde{\delta}$

$$\hat{Q} = e^{-\tilde{R}}(\tilde{\delta} + \tilde{Q})e^{\tilde{R}}, \qquad \tilde{\delta}\tilde{Q} = 0$$

$$\tilde{\delta} = \frac{1}{2}\lambda_{IJ}p_{\tilde{I}\tilde{J}} \qquad (\textstyle\int [dz] \text{ will be omitted})$$

$$\begin{aligned}\tilde{Q} &= -\lambda_+(p_- + 2i\partial x_I^+\theta_{\tilde{I}}) - \lambda_{\tilde{I}}(p_I + 2i\partial x_I^+\theta_+) \\ &\quad -ib_I\lambda_+\lambda_{\tilde{I}} - 4c_I\partial x_I^+ \quad \text{(without the quartet)}\end{aligned}$$

$$\tilde{R} = \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_5 + \tilde{R}_6 + \tilde{R}_8 + \tilde{R}_9$$

$$\begin{aligned}
\tilde{R}_2 &= -4\,i\lambda_+^{-1}\theta_+ c_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + \theta_+\theta_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + 4\lambda_+^{-2}c_{\tilde{I}}c_{\tilde{J}}\theta_{IJ}\partial\theta_+ \\
&\quad +4\,i\lambda_+^{-1}c_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - \theta_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - 4\lambda_+^{-1}c_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}} \\
&\quad -2\,i\theta_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}} \\
\tilde{R}_3 &= -\frac{i}{8}\epsilon_{IJKLM}\lambda_{LM}b_I\theta_{JK} \\
\tilde{R}_5 &= \frac{1}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_+\theta_{IN}\theta_{LM}\partial\theta_{\tilde{N}} - \frac{i}{6}\lambda_+^{-2}\epsilon_{IJKLM}\lambda_{JK}c_{\tilde{N}}\theta_{IN}\theta_{LM}\partial\theta_+ \\
&\quad +\frac{1}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_{\tilde{N}}\theta_{IN}\theta_{LM}\partial\theta_+ - \frac{i}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_{IN}\theta_{LM}\partial x^-_{\tilde{N}} \\
\tilde{R}_6 &= \frac{i}{3}\lambda_+^{-1}\epsilon_{IJKLM}c_{\tilde{N}}\theta_{IJ}\theta_{MN}\partial\theta_{KL} - \frac{1}{12}\epsilon_{IJKLM}\theta_{\tilde{N}}\theta_{IJ}\theta_{MN}\partial\theta_{KL} \\
\tilde{R}_8 &= \frac{1}{480}\lambda_+^{-2}\epsilon_{IJKLM}\epsilon_{NPQRS}\lambda_{LM}\lambda_{NS}\theta_{IP}\theta_{JK}\theta_{QR}\partial\theta_+ \\
\tilde{R}_9 &= \frac{1}{240}\lambda_+^{-1}\epsilon_{IJKLM}\epsilon_{NPQRS}\lambda_{JM}\theta_{IN}\theta_{KL}\theta_{PQ}\partial\theta_{RS}
\end{aligned}$$

2. Decoupling of $\{\lambda_{\tilde{I}}, \omega_I, c_I, b_I\}$ = quartet w.r.t $\bar{\delta}$

$$\tilde{Q} = e^{-S}(\bar{\delta} + \bar{Q})e^{S}\,, \qquad \bar{\delta}\bar{Q} = 0$$
$$\bar{\delta} = -ib_I\lambda_+\lambda_{\tilde{I}}\,, \qquad S = -i\lambda_+^{-1}c_I(p_I + 2i\partial x_I^+\theta_+)$$

$\hat{Q}$ reduced to

$$\bar{Q} = -\lambda_+ p_- - 2i\lambda_+\partial x_I^+\theta_{\tilde{I}}$$

We thus find

Cohomology of $\hat{Q}$ $\Leftrightarrow$ Cohomology of $\bar{Q}$ with two quartets removed

☐ **Simplifying BRST operator on RNS side**:

$$Q_{RNS} = \int [dz] \Big[cT_M - \frac{1}{2} e^{\phi}\eta G_M + bc\partial c$$
$$- c\left(\frac{1}{2}(\partial\phi)^2 + \partial^2\phi + \eta\partial\xi\right) - \frac{1}{4}e^{2\phi} b\eta\partial\eta \Big]$$

$$\begin{aligned} T_M &= -\frac{1}{2}(\partial x^\mu)(\partial x_\mu) - \frac{1}{2}\psi^\mu \partial\psi_\mu \\ G_M &= i\partial x^\mu \psi_\mu \\ \beta &= \partial\xi\, e^{-\phi}, \qquad \gamma = e^{\phi}\,\eta \end{aligned}$$

Berkovits showed:

Cohomology of Q_{RNS} in $\mathcal{H}_S$ (without ξ_0)	=	Cohomology of $Q'_{RNS} = \eta_0 + Q_{RNS}$ in $\mathcal{H}_L$ (with ξ_0) with finite range of "pictures"

Again introduce judicious degrees and apply the logic as on EPS side.

Omitting the details, we find ($\psi_I^- \equiv e_I^{-\mu}\psi_\mu$, etc.)

Sim. trans. of $\eta_0 + Q_{RNS}$

$$e^{-X}e^{Y}e^{X}e^{T}(\eta_0 + Q_{RNS})e^{-T}e^{-X}e^{-Y}e^{X}$$
$$= \eta_0 + Q_0 = \eta_0 + 4ice^{-\phi}\psi_I^- \partial x_I^+$$

where

$$\begin{aligned} T &= \int [dz] 2c\xi e^{-\phi} G_M^+ \\ X &= \int [dz] \left((4\psi_I^- \psi_I^+ - 2\partial\phi) c\partial c \xi \partial \xi \right) \\ Y &= \int [dz] \left(-\frac{i}{20} e^{3\phi} \psi_I^+ x_I^- b\partial b \eta \partial \eta - \frac{1}{5} e^{2\phi} b\eta (\psi_I^+ x_I^-)(\psi_J^+ \partial x_J^-) \right) \end{aligned}$$

□ Comparison between RNS and EPS:

By similarity transformations we obtained (excluding decoupled quartets in EPS)

$$\hat{Q} \quad \Rightarrow \quad \bar{Q} = -\lambda_+ p_- - 2i\lambda_+ \theta_{\tilde{I}} \partial x_I^+$$

$$\eta_0 + Q_{RNS} \quad \Rightarrow \quad \eta_0 + Q_0 = \eta_0 + 4ice^{-\phi}\psi_{\tilde{I}}^- \partial x_I^+$$

Correspondence between RNS and EPS fields

One can express λ_+ and its conjugate ω_- by introducing conjugate bosons (s, t)

$$\lambda_+ = e^s\,, \qquad \omega_- = \left(\frac{1}{2}\partial s + \partial t\right) e^{-s}\,, \quad s(z)t(w) = \ln(z-w)\,,\ s(z)s(w) = t(z)t(w) = 0$$

Then the following identification of fields holds in all respects (OPE, dimensions, etc.):

Field correspondence

$$-e^s p_- = \eta\,, \qquad e^{-s}\theta_+ = \xi$$

$$e^t \theta_+ = c\,, \qquad e^{-t} p_- = -b$$

$$e^{-s} p_I = -be^{\phi}\psi_I^+\,, \qquad e^s \theta_{\tilde{I}} = -2ce^{-\phi}\psi_{\tilde{I}}^-$$

This shows

$$\eta_0 + Q_0 = \eta_0 + 4ice^{-\phi}\psi^-_{\tilde{I}}\partial x^+_I = -\lambda_+ p_- - 2i\lambda_+\partial x^+_I\theta_{\tilde{I}} = \bar{Q}$$

$$\boxed{\hat{Q} \sim \eta_0 + Q_{RNS} \quad \text{cohomologically}}$$

□ **Proper Hilbert space and non-trivial cohomology**:

Inportant to clarify the proper Hilber space in which to consider the cohomology.

Generic Problem: Suppose $^\exists$ a fermionic operator $\Xi(z)$ "inverse" to the BRST operator Q in the sense

$$Q\Xi(z) = 1$$

Then, any BRST-closed operator $V(z)$ is BRST-exact, *i.e.* cohomology becomes trivial, since

$$Q(\Xi(z)V(z)) = (\underbrace{Q\Xi(z)}_{1})V(z) - \Xi(z)(\underbrace{QV(z)}_{0}) = V(z)$$

Problem in (E)PS: There indeed exists such an operator in (E)PS:

$$\Xi = \lambda_+^{-1}\theta_+ \,, \qquad \hat{Q} = -\lambda_+ p_- + \cdots$$

Resolution (Berkovits):

One can restrict the Hilbert space to **ASPC** (almost super-Poincaré covariant) subspace, where $\Xi = \lambda_+^{-1}\theta_+$ is excluded $\Rightarrow$ non-trivial cohomology

How is this justified on the RNS side ?

- Operators inverse to η_0 and Q_{RNS} separately are well-known:

$$\eta_0 \xi(z) = 1$$
$$Q_{RNS} Z(z) = 1 \,, \qquad Z = -4c\xi\partial\xi e^{-2\phi}$$

  But the one inverse to $Q'_{RNS} = \eta_0 + Q_{RNS}$ has not been identified.

- Cohomology of RNS is non-trivial in finite picture range $\Rightarrow$ The inverse, if it exists, must carry arbitrarily large picture.

- At the juncture of connecting RNS and EPS, Ξ is precisely the ξ ghost.
  Transform back to the original $\mathcal{H}_l$ of RNS by the inverse similarity transformation $\tilde{\xi} \equiv U^{-1}\xi U$. This indeed contains infinite range of pictures and should be disallowed.

4 Summary and Discussions

- Berkovits' pure spinor formalism is a promising new scheme for superstring, with manifest spacetime SUSY and Lorentz-covariant rules. It has had many successes already.
- However, its origin is unclear and produces various questions.
- Some of these problems can be overcome if PS constraints are removed.

$\Downarrow$

□ **EPS formalism**:

- All the constraints are removed with minimum number of additional ghosts $(c_I, b_I)_{I=1\sim 5}$. Should be useful for eventual "quantization".
- Bona fide nilpotent "BRST" charge is constructed, with powerful use of homological perturbation scheme, with correct cohomology.
- "b-ghost" is constructed and the relation to the Virasoro operator is clarified. Twisted $N = 2$ superconformal structure is identified.
- Massless vertex operators are constructed.

- Relation to RNS formalism is explicitly established by a similarity transformation.

Remaining problems:

- More complete understanding of the relation to RNS
  - Zero mode structure.
- Derivation of the zero-mode rule and computation of the scattering amplitudes.
- Loop amplitudes
- Coupling to non-trivial backgrounds
- Applications to
  - Superparticle
  - Supermembrane
  - D-brane
  - $\mathbf{AdS}_5 \times \boldsymbol{S}^5$ background

- Completely covarint formulation. (Recent proposal by Stony Brook group based on the original (second-class) algebra of superinvariants is interesting but yet rather unsystematic and complicated.)
- **Search for the underlying action, clarification of its local symmetries and quantization**

- **How should we think of λ^α ?**

– Bosonic ghosts ? Hard to understand the PS constraints.

– Worldsheet SUSY partner of θ^α in doubly supersymmetric formulation ? How to remove fields with half-integral spin ?

– A kind of harmonic variable ?

- Must find appropriate **local fermionic symmetry**, which may be more generic than conformal symmetry.

Expect many surprises in the future !