Loop equation, matrix model and string field theory

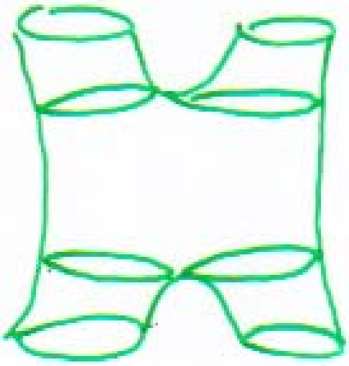
H. Kawai

based on collaborations with

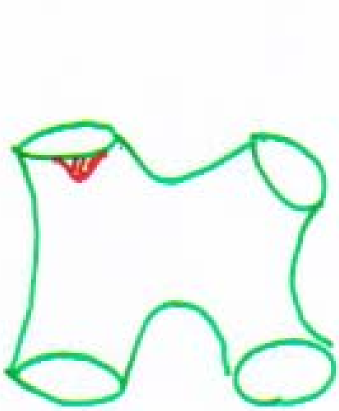
- Hayakawa, Ishibashi and Tada
- Hanada, Kuroki and Matsuo

0. Introduction

String field theory is unnatural, in the sense that artificial way of cutting the worldsheet is needed. We need to decompose the worldsheet into propagators and interactions.



Loop equation is natural. We consider deformation of loops, and need not to introduce an artificial decomposition.



Loop equation is more fundamental than string field.

Is loop equation complete?
Can it describe the non-perturbative effects?

1. Instanton in c=0 noncritical string

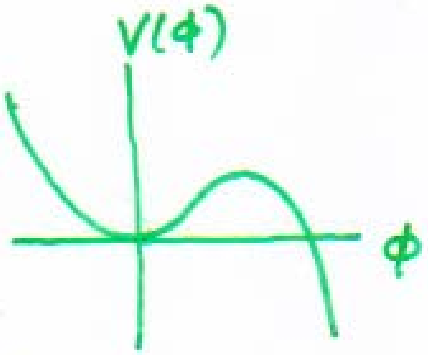
At present we have a complete set of loop equations for noncritical string, and not for the critical string.

We consider $c=0$ noncritical string.

one-matrix model

$$S = N \,\mathrm{tr}\, V(\phi)$$

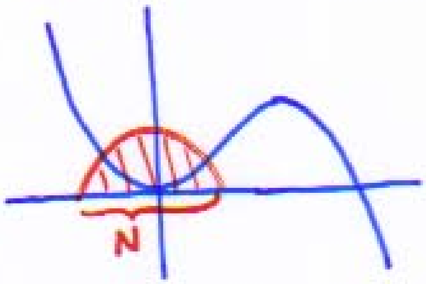
ϕ : $N \times N$ hermitian



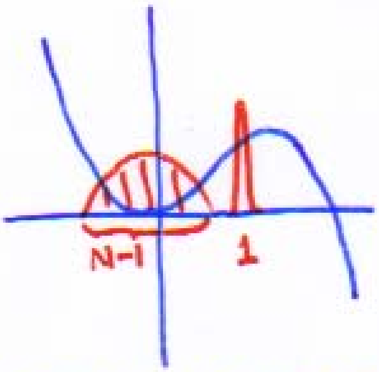
effective action for the eigenvalues

$$S_{eff} = -\sum_{i<j} \log(\lambda_i - \lambda_j)^2 + \sum_i N\, V(\lambda_i)$$

eigenvalue distribution



"ground state"



"excited state"

effective potential for the N-th eigenvalue

$$V_{eff}(\lambda_N) = -\sum_{i=1}^{N-1} \log(\lambda_N - \lambda_i)^2 + N V(\lambda_N)$$

the force on λ_N

$$-\frac{\partial}{\partial \lambda_N} V_{eff}(\lambda_N) = \sum_{i=1}^{N-1} \frac{2}{\lambda_N - \lambda_i} - N V'(\lambda_N)$$

$$\sim O(N)$$

up to $O(1)$ correction

$$-\frac{\partial}{\partial \lambda} V_{eff}(\lambda) = N\left(2 \mathrm{Re}\, R(\lambda + i0) - V'(\lambda)\right)$$

$$R(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \lambda_i} \quad : \text{resolvent}$$

$$\mathrm{Re}\, R(\lambda + i0) = \int d\lambda' \, \mathrm{PV} \frac{\rho(\lambda')}{\lambda - \lambda'}$$

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

loop equation for the resolvent

$$0 = \int d^N\lambda \sum_{i=1}^{N} \frac{\partial}{\partial\lambda_i} \frac{1}{\lambda-\lambda_i} e^{-S_{eff}}$$

$$= \int d^N\lambda \left\{ \sum_{i=1}^{N} \frac{1}{(\lambda-\lambda_i)^2} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{1}{\lambda-\lambda_i} \frac{2}{\lambda_i-\lambda_j} - N \sum_{i=1}^{N} \frac{V'(\lambda_i)}{\lambda-\lambda_i} \right\} e^{-S_{eff}}$$

$$\left(\sum_{i\neq j} \frac{1}{\lambda-\lambda_i} \frac{2}{\lambda_i-\lambda_j} = \sum_{i\neq j} \left(\frac{1}{\lambda-\lambda_i} - \frac{1}{\lambda-\lambda_j} \right) \frac{1}{\lambda_i-\lambda_j} = \sum_{i\neq j} \frac{1}{(\lambda-\lambda_i)(\lambda-\lambda_j)} \right)$$

$$= \int d^N\lambda \left\{ \left(\sum_{i=1}^{N} \frac{1}{\lambda-\lambda_i} \right)^2 - N \sum_{i=1}^{N} \frac{V'(\lambda_i)}{\lambda-\lambda_i} \right\} e^{-S_{eff}}$$

$$\left(\frac{V'(\lambda_i)}{\lambda-\lambda_i} = \frac{V'(\lambda_i)-V'(\lambda)}{\lambda-\lambda_i} + \frac{1}{\lambda-\lambda_i} V'(\lambda) \right)$$

$$\Rightarrow \langle R(\lambda)^2 \rangle - V'(\lambda) \langle R(\lambda) \rangle = -\frac{1}{N} \left\langle \sum_i \frac{V'(\lambda)-V'(\lambda_i)}{\lambda-\lambda_i} \right\rangle$$

$$= f(\lambda)$$

$$V(x) = \frac{x^2}{2} - g\frac{x^3}{3} \Rightarrow V' = x - gx^2$$

$$f(x) = \frac{1}{N} \left\langle \sum_i \left(-1 + g(x+\lambda_i) \right) \right\rangle$$

$$= gx + \left(-1 + g \frac{1}{N} \left\langle \sum_{i=1}^{N} \lambda_i \right\rangle \right)$$

$$= gx + a$$

a is fixed by the singularity structure of R.

large-N factorization

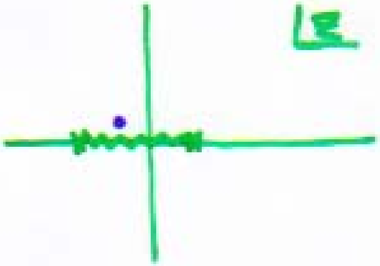
$$R(\lambda)^2 - V'(\lambda)\ R(\lambda) = f(\lambda)$$

$$R(\lambda) = \frac{1}{2}\left\{ V'(\lambda) + \sqrt{V'(\lambda)^2 + 4f(\lambda)} \right\}$$

Singularity of $R(z)$

$$R(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i}$$

λ_i : real



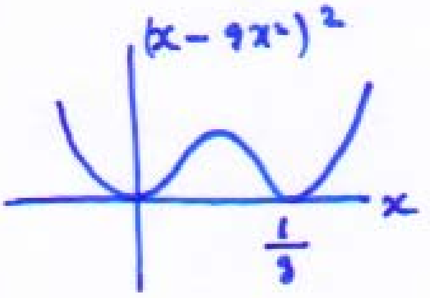
(i) cut should be on the real axis.

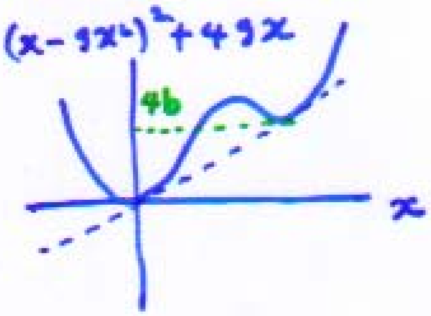
$$R(\lambda+i0) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda+i0-\lambda_i}$$
$$= \frac{1}{N}\sum_{i=1}^{N} PV\frac{1}{\lambda-\lambda_i} - i\pi\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$$

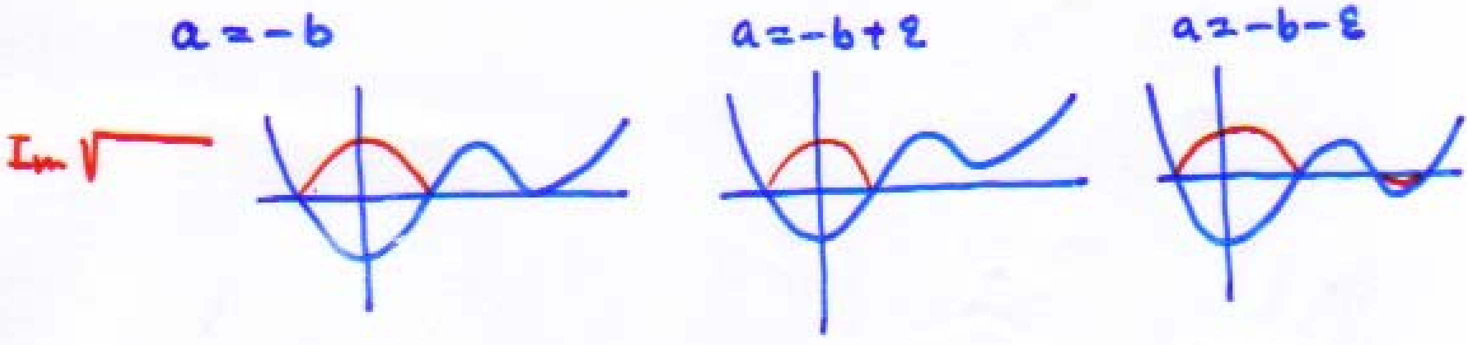
(ii) $\mathrm{Im}\, R(\lambda+i0) < 0$

$$V(x) = \frac{x^2}{2} - \frac{g}{3}x^3$$

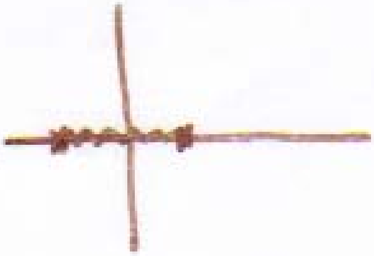
$$V'(x)^2 + 4f(x) = (x-gx^2)^2 + 4gx + 4a$$

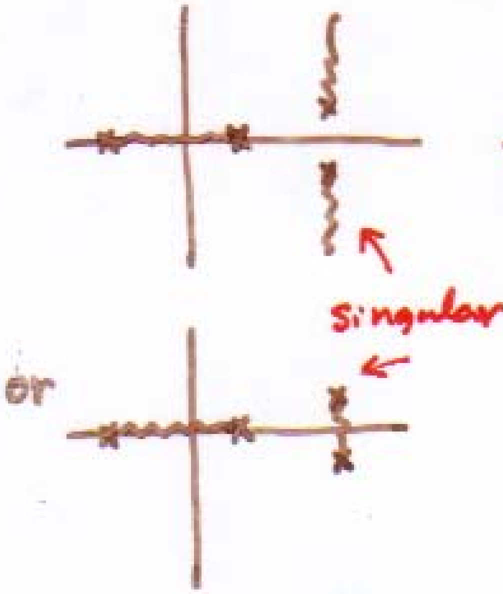


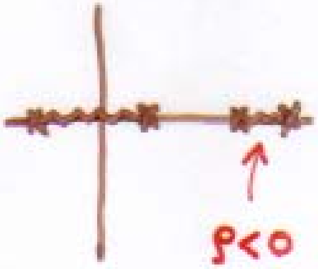




singularity of $R(\theta)$





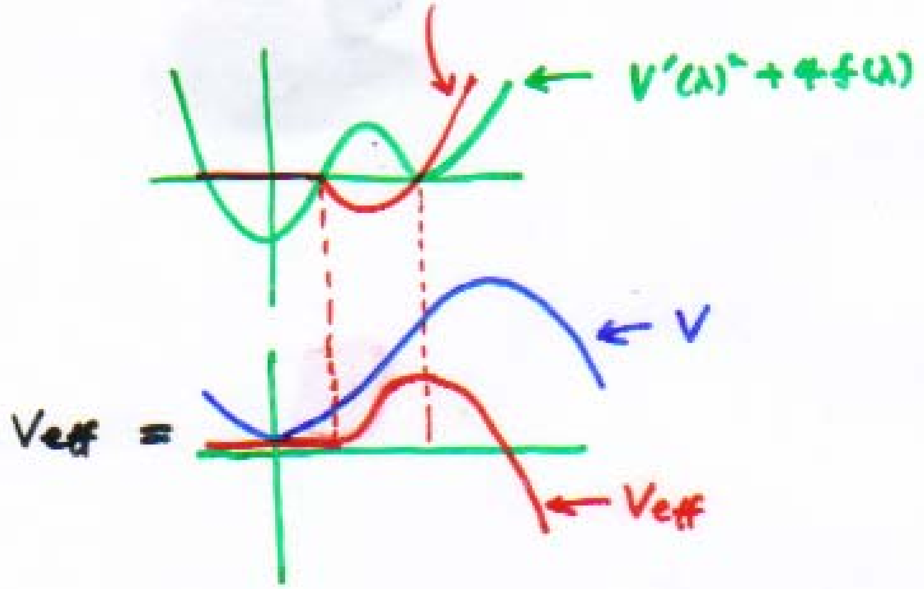


Only the one cut solution $a=-b$ satisfies (i) and (ii).

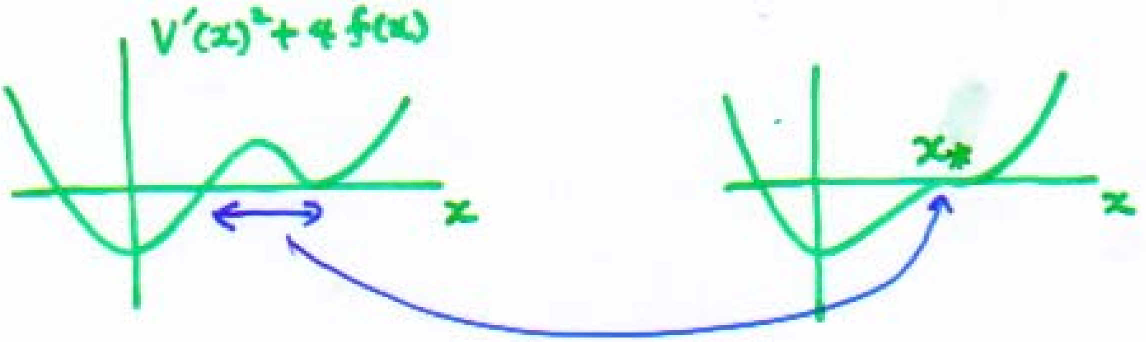
Let's return to $V_{eff}(\lambda)$.

$$-\frac{\partial}{\partial\lambda} V_{eff}(\lambda) = N\left(2\,\mathrm{Re}\,R(\lambda+i0) - V'(\lambda)\right)$$

$$= N\,\mathrm{Re}\sqrt{V'(\lambda)^2 + 4f(\lambda)}$$



near the critical point



$g < g_*$ $g = g_*$

$$V'(x)^2 + 4f(x) = (x - x_*)^3 + (g - g_*)(x - x_*) + \mathrm{const}$$

$$= (x - x_*)^3 + \frac{3}{4}(g - g_*)(x - x_*) + (g_* - g)^{\frac{3}{2}}$$

$$= \left(y - \frac{1}{2}\sqrt{\Delta g}\right)^2 \left(y + \sqrt{\Delta g}\right)$$

$$y = x - x_*$$

$$\Delta g = g_* - g$$

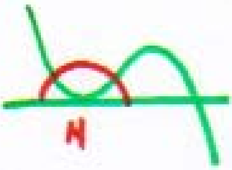
action of the instanton

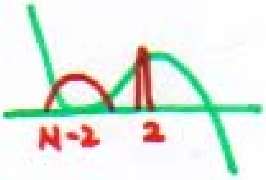
$$\frac{\partial}{\partial y} V_{eff} = -N\left(y - \frac{1}{2}\sqrt{\Delta g}\right)\sqrt{y+\sqrt{\Delta g}}$$

$$\Delta S = -N\int_{-\sqrt{\Delta g}}^{\frac{1}{2}\sqrt{\Delta g}} dy \ \left(y - \frac{1}{2}\sqrt{\Delta g}\right)\sqrt{y+\sqrt{\Delta g}}$$

$$= c\, N (\Delta g)^{\frac{5}{4}} \quad , \ c = \frac{3}{10}\sqrt{6}$$

$$Z = e^{-N^2 E_0}\left(1 + e^{-cN(\Delta g)^{\frac{5}{4}}} + \frac{1}{2} e^{-2cN(\Delta g)^{\frac{5}{4}}} + \cdots\right)$$





$$= e^{-N^2 E_0} \cdot e^{e^{-cN(\Delta g)^{\frac{5}{4}}}}$$

$$= e^{-N^2 E_0 + e^{-cN(\Delta g)^{\frac{5}{4}}}}$$

interaction between instantons is negligible.

$$S_{eff} = -\sum_{1\leq i\leq j\leq N-2} \log(\lambda_i - \lambda_j)^2 + N\sum_{i=1}^{N-2} V(\lambda_i)$$

$$- \sum_{1\leq i\leq N-2} \log(\lambda_{N-1} - \lambda_i)^2 + N\, V(\lambda_{N-1}) \leftarrow O(N)$$

$$- \sum \log(\lambda_N - \lambda_i)^2 + N\, V(\lambda_N) \leftarrow O(N)$$

$$- \log(\lambda_{N-1} - \lambda_N)^2 \qquad \leftarrow O(1)$$

negligible

double scaling limit

$$E_0 \propto (\Delta g)^{\frac{5}{2}}$$

$$N^2 E_0 = N^2 (\Delta g)^{\frac{5}{2}} = t^{\frac{5}{2}}$$

instanton action $\quad N(\Delta g)^{\frac{5}{4}} = t^{\frac{5}{4}}$

Instanton is finite in the double scaling limit:

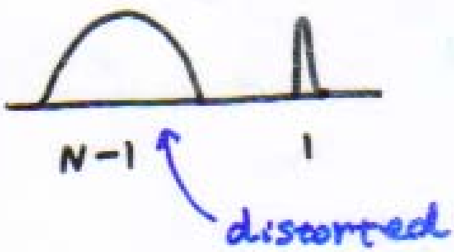
$$Z = e^{-(t^{\frac{5}{2}} + \cdots)} + e^{-ct^{\frac{5}{4}}}$$

$$F = t^{\frac{5}{2}} + t^0 + t^{-\frac{5}{2}} + \cdots \qquad - e^{-ct^{\frac{5}{4}}}$$



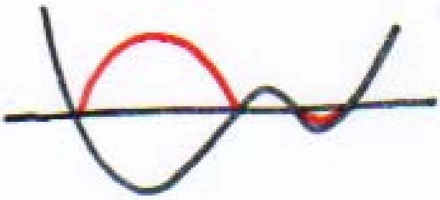
instantons

back reaction to the other eigenvalues



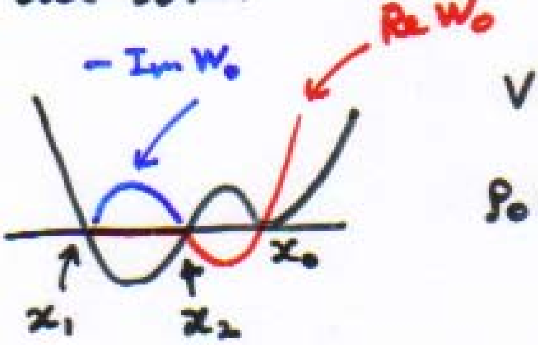
The N-th eigenvalue distorts the distribution of the other eigenvalues.

$$\Uparrow$$



2-cut solution

one-cut solution



$$V'(x)^2 + 4 f_0(x) = W_0(x)^2$$

$$\rho_0(x) = -\frac{1}{\pi} \mathrm{Im}\, R(x+i0)$$

$$= -\frac{1}{2\pi} \mathrm{Im}\, W_0(x+i0)$$

two-cut solution

$$V'(x)^2 + 4 f_0(x) - \alpha = W(x)^2$$

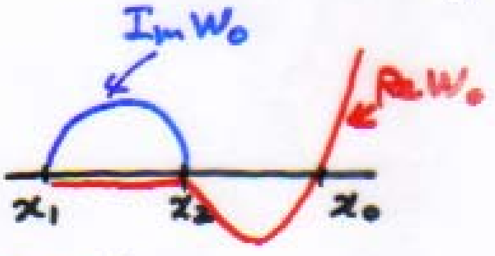
$$\rho(x) = -\frac{1}{2\pi} \mathrm{Im}\, W(x+i0)$$

$$= -\frac{1}{2\pi} \mathrm{Im} \sqrt{W_0(x+i0)^2 - \alpha}$$

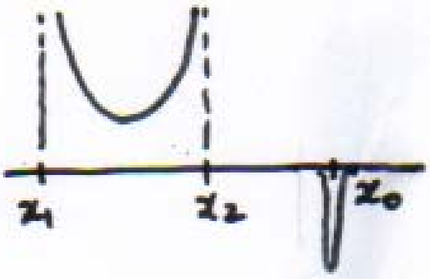
$$= -\frac{1}{2\pi} \mathrm{Im}\, W_0 + \frac{\alpha}{2\pi} \mathrm{Im} \frac{1}{2W_0} + O(\alpha^2)$$

$$= \rho_0(x) + \delta\rho(x)$$

$$\delta\rho(x) = \frac{\alpha}{2\pi}\, \mathrm{Im} \frac{1}{2W_0(x+i0)}$$



$$= \alpha \left\{ \frac{1}{(2\pi)^2} \frac{1}{\rho_0(x)}\, \theta(x_1 \le x \le x_2) - \frac{1}{2}\frac{1}{W_0'(x_0)}\, \delta(x-x_0) \right\}$$



If we choose $\alpha = -2W_0'(x_0)\frac{1}{N}$,

we formally obtain $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx\ \delta\rho(x) = \frac{1}{N}$,

which is the right normalization for one instanton.

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)$$

The distortion of $\rho(x)$ is given by

$$\delta\rho(x) = \frac{1}{N}\frac{-2}{(2\pi)^2} W'(x_0)\frac{1}{\rho_0(x)}\,\theta(x_1\le x\le x_2) + \frac{1}{N}\delta(x-x_0)$$

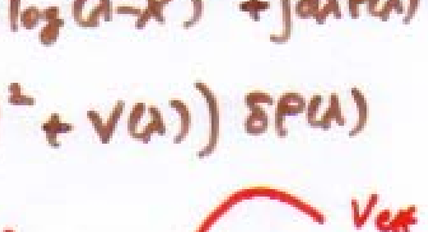
action of the instanton is given by

$$\frac{1}{N^2}\delta S_{eff} = \delta\left(-\frac{1}{2}\int d\lambda d\lambda'\, \rho(\lambda)\rho(\lambda')\log(\lambda-\lambda')^2 + \int d\lambda\, \rho(\lambda)V(\lambda)\right)$$

$$= \int d\lambda\left(-\int d\lambda'\rho(\lambda')\log(\lambda-\lambda')^2 + V(\lambda)\right)\delta\rho(\lambda)$$

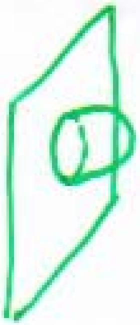
$$= \int d\lambda\ V_{eff}(\lambda)\,\delta\rho(\lambda)$$

$$= \frac{1}{N}\ V_{eff}(x_0)$$



2. Instanton as a disk

- D-brane



In the ordinary critical string, D-brane action is given by a disk.

- c=0 noncritical string

one-instanton effect

$$S_{eff} = -\sum_{1\le i<j\le N-1} \log(\lambda_i-\lambda_j)^2 + N\sum_{i=1}^{N-1} V(\lambda_i) \quad \leftarrow N-1$$

$$-\sum_{i=1}^{N-1} \log(\lambda-\lambda_i)^2 + N\,V(\lambda) \quad \leftarrow 1$$

This is equivalent to the following action:

$$S^{(1)} = N\,\mathrm{tr}\,V(\phi') + q^{\dagger}(\phi'-\lambda)q + N\,V(\lambda)$$

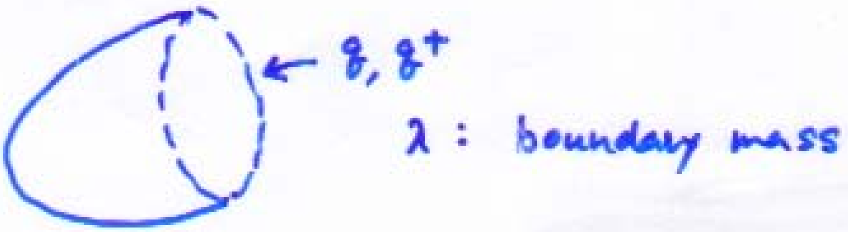
ϕ' : $(N-1)\times(N-1)$ hermitian

q : fundamental rep. of $SU(N-1)$
Grassmann odd

λ : real number

- $SU(N-1) \Rightarrow \phi' = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_{N-1}\end{pmatrix}$
- $\int d^{N-1}q\, d^{N-1}q^{\dagger} \Rightarrow \prod_{i=1}^{N-1} (\lambda_i-\lambda)^2$
- $N\,V(\lambda)$

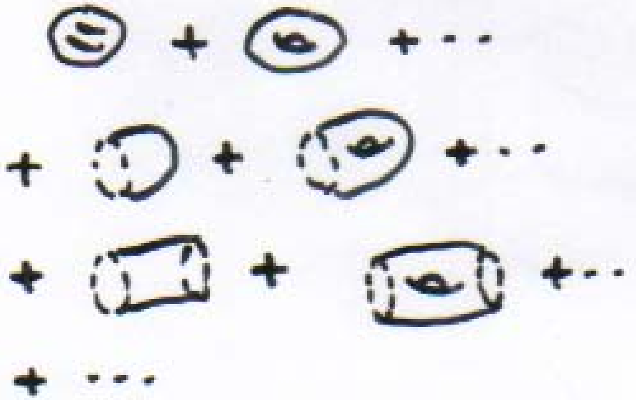
For fixed λ, q, q^+ integration gives open boundaries.



λ : boundary mass

$$Z = \int d\lambda \, e^{-NV(\lambda)} \, e^{-F}$$

F =



This is one-instanton effect.

two-instanton effect

$$S^{(2)} = N \,\mathrm{tr}\, V(\phi') + q_1^+(\phi'-\lambda_1) q_1 + N V(\lambda_1)$$
$$+ q_2^+(\phi'-\lambda_2) q_2 + N V(\lambda_2)$$
$$- \log(\lambda_1-\lambda_2)^2$$

can be neglected in the leading order of $N \to \infty$.

ϕ' : $(N-2) \times (N-2)$ hermitian

q_1, q_2 : $SU(N-2)$ fundamental rep.

λ_1, λ_2 : real

$$\phi = \left(\begin{array}{c|cc} \overbrace{\phi'}^{N-2} & & \\ \hline & \lambda_1 & \\ & & \lambda_2 \end{array}\right)$$

k-instanton effect

$$S^{(k)} = N \operatorname{tr} V(\phi') + \sum_{i=1}^{k} q_i^{+} \phi' q_i - \sum_{i,j=1}^{N} q_i^{+} q_j \lambda_{ij}$$
$$+ N \operatorname{tr} V(\lambda)$$

ϕ' : $(N-k) \times (N-k)$ hermitian

λ : $k \times k$ hermitian

$q_1, \cdots, q_k$: $SU(N-k)$ fundamental

$$\phi = \left(\begin{array}{c|c} \phi' & \\ \hline & \lambda \end{array}\right)$$
$$\underbrace{\hphantom{\phi'\phi'}}_{N-k}\;\underbrace{\hphantom{\lambda\lambda}}_{k}$$

SU(k) looks like flavor.

If $k \ll N$, the effect of the van der Mont det for λ can be neglected, and the k-instanton effect is equal to $\frac{1}{k!}(\text{one-instanton})^k$.

Therefore, the multi instanton effect can be expressed as

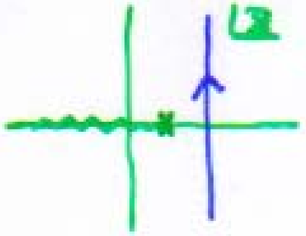
$$Z = e^{-(\ominus + \circledcirc + \cdots)} \, e^{\int d\lambda \, e^{-NV(\lambda)} e^{-(\odot + \circledcirc \cdots)}}$$

comparison to Liouville theory

loop amplitude

$$\Psi(\ell) = \frac{1}{N} \operatorname{tr} e^{\ell\phi} = \frac{1}{N}\sum_{i=1}^{N} e^{\ell\lambda_i}$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\, e^{\ell z}\, \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z-\lambda_i}$$



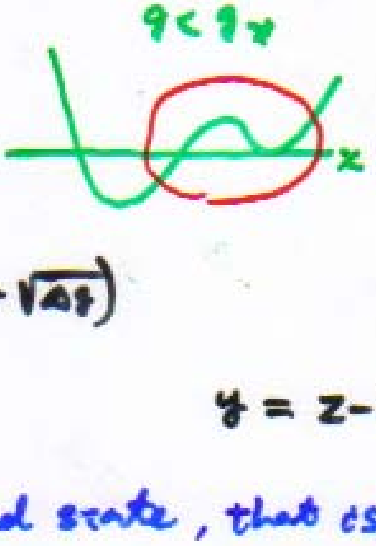
$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\, e^{\ell z}\, R(z)$$

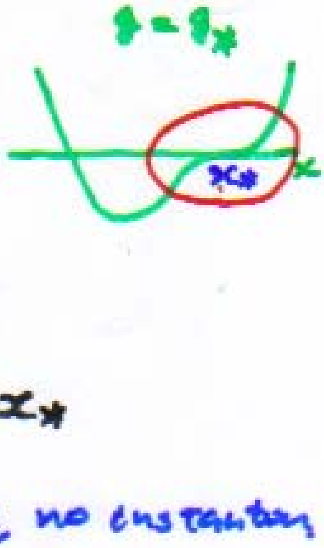
$$R(z) = \frac{1}{2}\left\{ V'(z) + \sqrt{V'(z)^2 + 4f(z)} \right\}$$

$$= \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\, e^{\ell z}\, W(z)$$

$$W(z) = \frac{1}{2}\sqrt{V'(z)^2 + 4f(z)}$$

near the critical point





$\boxed{\text{ground state}}$

$$W_0(z)^2 = \frac{1}{4}V'(z)^2 + f_0(z)$$

$$= \left(y - \frac{1}{2}\sqrt{\Delta g}\right)^2 \left(y + \sqrt{\Delta g}\right)$$

$$= w_0(y)^2 \qquad\qquad y = z - x_*$$

0 stands for the ground state, that is, no instanton

The upper case for the original matrix model.

The lower case for the continuum limit.

$$\Psi_0(\ell) = \int_{c-i\infty}^{c+i\infty} \frac{dy}{2\pi i}\, e^{\ell x_*}\, e^{\ell y}\, w_0(y)$$

$$= e^{\ell x_*} \longleftarrow \text{renormalization of the boundary cosmological const.}$$
$$\quad\ \ \psi_0(\ell)$$

$$\psi_0(\ell) = \int_{c-i\infty}^{c+i\infty} \frac{dy}{2\pi i}\, e^{\ell y}\, w_0(y)$$

w_0 is Laplace tr. of ψ_0.

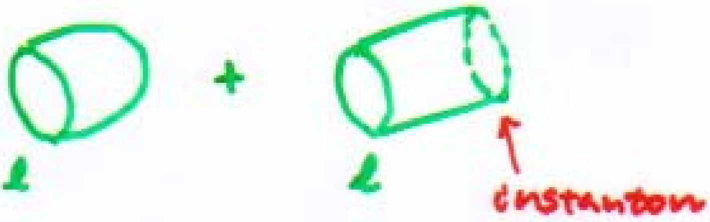
one-instanton state

$$W(z)^2 = W_0(z)^2 - \alpha \quad , \quad \alpha = -\frac{1}{N}\cdot\text{const}$$

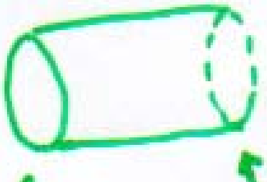
$$w(y)^2 = w_0(y)^2 - \alpha$$

$$w(y) = w_0(y) + \text{const}\,\underbrace{\frac{1}{N}\,\frac{1}{w_0(y)}}$$

distortion of the loop amplitude
due to the instanton.



Liouville theory



loop ↖ ZZ boundary state

$\int \frac{dl}{l} e^{-ly}$ $l \leftrightarrow y$

$\rightarrow |B_s\rangle$

$$|B\rangle_{ZZ} = |B_{i(b+\frac{1}{b})}\rangle - |B_{i(b-\frac{1}{b})}\rangle$$

$$e^{\pi b s} = y + \sqrt{y^2 - t}$$

$t = \Delta g$

$b = \sqrt{\frac{2}{3}}$

$$\int d\tau \ \langle B_s | e^{-\pi\tau(L_0 + \bar{L}_0)} |B\rangle_{ZZ}$$

$$= \int d\tau \ \eta(q)^2$$ ← ghost

$$\times \sqrt{2} \int_0^\infty dp \ \frac{q^{-p^2}}{\eta(q)} \cos(2\pi s p)$$ ← Liouville

$$= -\frac{1}{4} \log \frac{\cosh\left(\frac{2\pi s}{\sqrt{24}}\right) + \frac{\sqrt{3}}{2}}{\cosh\left(\frac{2\pi s}{\sqrt{24}}\right) - \frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{4} \log \frac{(\quad + \quad)^{\frac{1}{2}} + (\quad - \quad)^{\frac{1}{2}} + \sqrt{3}}{(y + \sqrt{y^2 - t})^{\frac{1}{2}} + (\quad - \quad)^{\frac{1}{2}} - \sqrt{3}}$$

↖ $w(y) = \int dl \ e^{-yl}$

$$= \frac{\partial}{\partial y}(\text{ the above }) = -\frac{\sqrt{6}}{8} \frac{1}{(y - \frac{1}{2}\sqrt{t})\sqrt{y + \sqrt{t}}}$$

$$= \text{const} \ \frac{1}{w_0(y)}$$

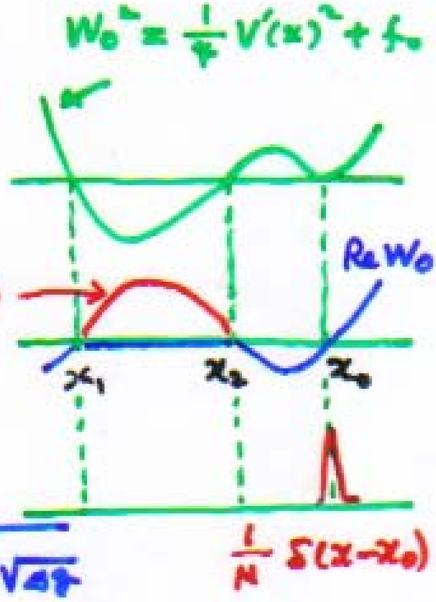
3. continuum loop equation

Is the instanton classical solution of the continuum loop equation ?

normalization

one-instanton effect on loop amplitude

$$W = \sqrt{W_0^2 - \alpha}$$
$$= W_0 - \frac{\alpha}{2W_0}$$
$$= W_0 + \frac{1}{N} W_0'(x_0) \frac{1}{W_0(x_0)}$$



near the critical point

$$W_0 \simeq \left((x-x_*) - \frac{1}{2}\sqrt{\Delta g}\right)\sqrt{(x-x_*)+\sqrt{\Delta g}}$$

double scaling limit

$$\begin{cases} \Delta g = a^2 t \\ x - x_* = a\xi \quad , \quad a \to 0 \\ N = a^{-5} \end{cases}$$

$$W_0 = a^{\frac{3}{2}} \overbrace{\left(\xi - \frac{1}{2}\sqrt{t}\right)\sqrt{\xi + \sqrt{t}}}^{w_0(\xi)}$$

$$W_0'(x_0) = a^{\frac{3}{2}} \cdot \frac{1}{a}\sqrt{\frac{3}{2}\sqrt{t}}$$

$x = x_0 \Leftrightarrow \xi = \frac{1}{2}\sqrt{t}$

$\frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial \xi}$

$$W = a^{\frac{3}{2}} w_0(\xi) + \frac{1}{N}\frac{1}{a}\sqrt{\frac{3}{2}\sqrt{t}}\ \frac{1}{w_0(\xi)}$$
$$= a^{\frac{3}{2}} \left\{ w_0(\xi) + \underbrace{\frac{1}{N}\frac{1}{a^{\frac{5}{2}}}}_{1}\sqrt{\frac{3}{2}\sqrt{t}}\ \frac{1}{w_0(\xi)} \right\}$$
$$= a^{\frac{3}{2}} \left\{ w_0(\xi) + \underbrace{\sqrt{\frac{3}{2}}\, t^{\frac{1}{4}}}_{\uparrow \ \text{finite}} \frac{1}{w_0(\xi)} \right\}$$

continuum loop equation

$$\ell \int_0^\ell d\ell' \ \langle \psi(\ell') \psi(\ell-\ell') \ \psi(\ell_1) \cdots \psi(\ell_n) \rangle$$

$$+ \ \rho(\ell) \ \langle \psi(\ell_1) \cdots \psi(\ell_n) \rangle$$

$$+ \ \sum_{i=1}^{n} \ \langle \psi(\ell+\ell_i) \ \psi(\ell_1) \ \underset{\text{remove } i\text{-th}}{\cdots\hat{}\cdots} \ \psi(\ell_n) \rangle$$

$$= 0$$

$$\rho(\ell) = -3 \, \delta''(\ell) + \frac{3}{4} t \, \delta(\ell)$$

disk amplitude

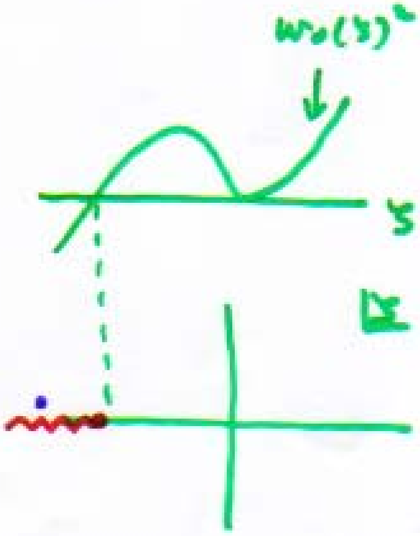
$$\langle \psi(\ell) \rangle_{disk} = \psi_0(\ell) = \int_{c-i\infty}^{c+i\infty} \frac{d\zeta}{2\pi i} \ e^{\ell\zeta} \ w_0(\zeta)$$

$$\ell \int_0^\ell d\ell' \ \psi_0(\ell') \psi_0(\ell-\ell') = -\rho(\ell)$$

$$\frac{\partial}{\partial\zeta} (w_0(\zeta))^2 = 3\zeta^2 - \frac{3}{4} t$$

$$w_0(\zeta)^2 = \zeta^3 - \frac{3}{4} \zeta t + \text{const}$$

$$= (\zeta + \sqrt{t})(\zeta - \frac{1}{2}\sqrt{t})^2$$

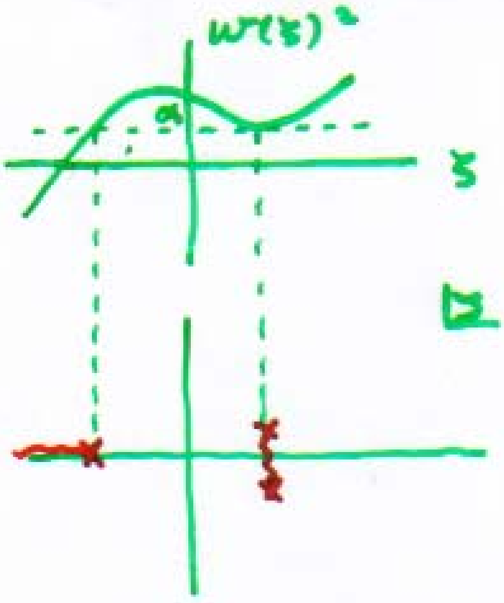


The ground state is obtained by solving the loop equation with the condition that the singularity of $w(\zeta)$ is only on the real axis and that $\mathrm{Im}\, w(x+i0) < 0$.

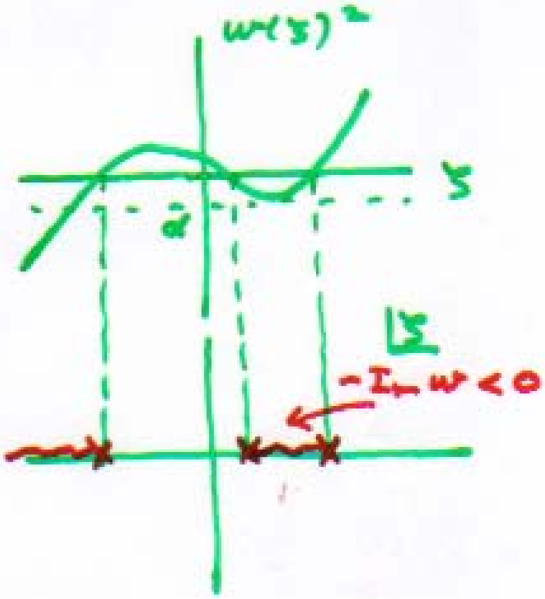
other classical solution ?

$$w(\zeta)^2 = (\zeta+\sqrt{t})(\zeta-\tfrac{1}{2}\sqrt{t})^2+\alpha$$

$$= w_0(\zeta)^2+\alpha$$



$\alpha > 0$



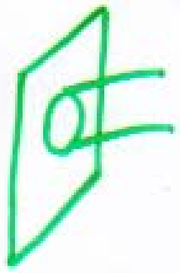
$\alpha < 0$

$$w(\zeta) = \sqrt{w_0(\zeta)^2+\alpha}$$

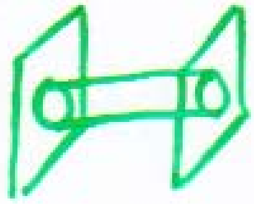
$$= w_0(\zeta) + \frac{\alpha}{2w_0(\zeta)} + \star\alpha^2 + \cdots$$

It seems α represents the number of instantons. However α is a real number and not quantized to an integer.

This situation is similar to the ordinary D-branes in the critical string theory.



In order to fix the overall normalization of D-branes, one should consider the dual channel, that is, the open string one-loop amplitude.



One common expectation is that if we consider the closed string field theory, the non-linearity of the equation fixes the normalization, such as

$$\phi^2 - \phi = 0 \quad \Rightarrow \quad \phi = 0 \text{ or } 1 .$$

However, it is not the case at least in the classical level of the noncritical string.

In our case, the eq. of motion is indeed nonlinear,

$$\frac{\partial}{\partial \zeta}(w(\zeta)^2) = p(\zeta) ,$$

but the parameter α comes in as an integration constant and not fixed by the eq. of motion.

4. continuum reformulation of $S^{(1)}$

$$S^{(1)} = N\,\mathrm{tr}\,V(\phi') + q^{+}(\phi'-\lambda)q + NV(\lambda)$$

ϕ' : $(N-1)\times(N-1)$ hermitian

q : $SU(N-1)$ fundamental

λ : real

$$\phi = \left(\begin{array}{c|c} \phi' & \\ \hline & \lambda \end{array}\right)$$

$$\underbrace{\qquad\quad}_{N-1}\ \underbrace{\quad}_{1}$$

effective potential for λ

$$Z = \int d\phi'\, dq\, dq^{+}\, d\lambda\ e^{-S^{(1)}}$$

$$= \int d\lambda\ e^{-N V_{eff}(\lambda)}$$

$$V_{eff}(x) = V(x) - \int^{x} dx'\,\mathrm{Re}\,R(x')$$

$$= -2\int^{x} dx'\,\mathrm{Re}\,W_0(x')$$

$$W_0(x) = \frac{1}{2}\sqrt{V'(x)^2 + 4f_0(x)}$$

$$= a^{\frac{3}{2}}\left(\zeta - \frac{1}{2}\sqrt{t}\right)\sqrt{\zeta+\sqrt{t}} = a^{\frac{3}{2}} w_0(\zeta)$$

$$N V_{eff}(\zeta) = -N\cdot 2\cdot a^{\frac{3}{2}}\cdot a\int^{\zeta} d\zeta'\ w_0(\zeta)$$

$x - x_* = a\zeta$

$$= -2\int^{\zeta} d\zeta'\, w_0(\zeta) \quad \leftarrow \quad N a^{\frac{5}{4}} = 1$$

$$= 2\int_0^{\infty}\frac{d\ell}{\ell}\, e^{-\ell\zeta}\, \underbrace{\Psi_0(\ell)}_{\substack{\uparrow \\ \text{loop amplitude}}}$$

In the continuum limit, we obtain

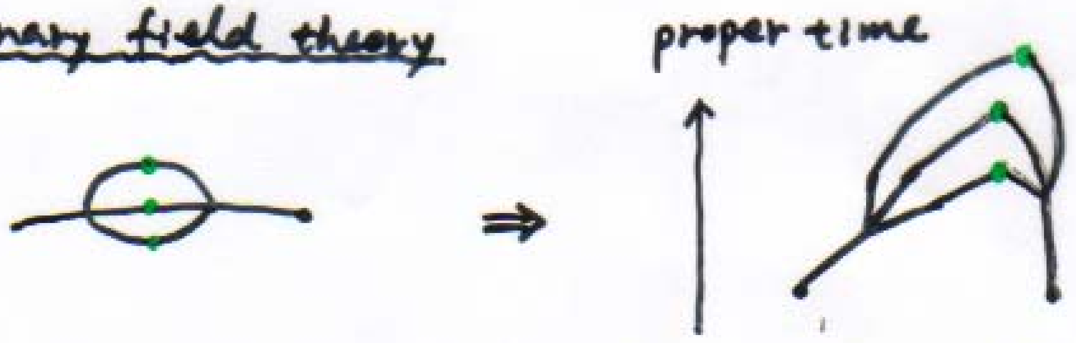
$$Z = \int d\phi' \int dm \ e^{-N \operatorname{tr} V(\phi') - 2\int_0^\infty \frac{dl}{l} e^{-lm} \psi(l)} .$$

ϕ' can be described by the ordinary continuum loop equation.

In order to analize how the extra term appeares in the loop equation, we use a special type of string field theory in which the relation to the loop equation is manifest.

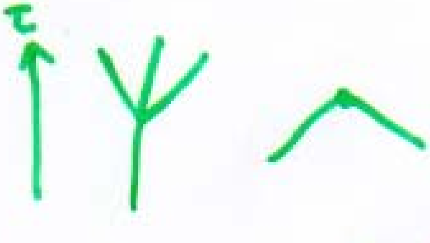
A special string field theory

ordinary field theory



We slice the Feynman diagram from the external legs along the proper time.

ex. ϕ^4-theory



$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z} \int d\phi \, \phi(x_1) \cdots e^{-S[\phi]}$$

$$= \lim_{\tau \to \infty} \langle 0 | e^{-\tau H} \phi^\dagger(x_1) \cdots \phi^\dagger(x_n) | 0 \rangle$$

$$[\phi(x), \phi^\dagger(y)] = \delta^D(x-y)$$

$$H = \int d^Dx \left\{ \frac{\delta S}{\delta \phi(x)}(\phi^\dagger) \, \phi(x) + \phi(x)^2 \right\}$$

ex. $$S = \int d^Dx \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right\}$$

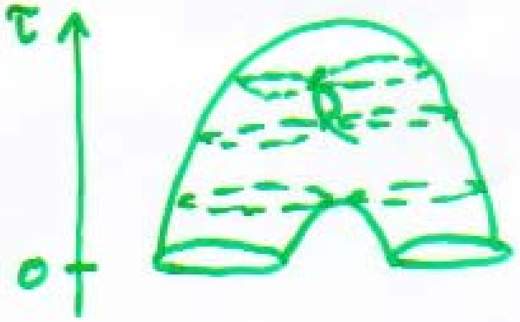
$$H = \int d^Dx \left\{ \left(-\Box \phi^\dagger + m^2 \phi^\dagger + \frac{\lambda}{3!} (\phi^\dagger)^3 \right) \phi(x) + \phi(x)^2 \right\}$$

⇔ Stochastic quantization

H : Fokker-Planck Hamiltonian

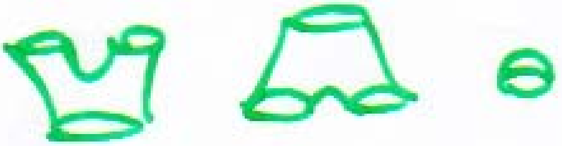
c=0 string

Ishibashi & H.K.,
Jevicki & Rodorigues



We sclice the worldsheet along the equi proper time curves.
Here we measure the proper time from the boundary loops.

elementary processes



$$H = \int dl_1 dl_2 \; \psi^+(l_1)\psi^+(l_2)\psi(l_1+l_2)(l_1+l_2) \qquad \infty \Rightarrow OO$$

$$+ \int dl_1 dl_2 \; \psi^+(l_1+l_2)\psi(l_1)\psi(l_2)\, l_1 l_2 \qquad OO \Rightarrow \infty$$

$$+ \int dl \; \rho(l)\psi(l) \qquad \theta \Rightarrow$$

$$[\psi(l), \psi^+(l')] = \delta(l-l')$$

$\psi(l)$: annihilation op. of a loop of length l
$\psi^+(l)$: creation " "

$$\rho(l) = -3\delta''(l) + \frac{3}{4} t\,\delta(l)$$

$$\langle \psi(l_1) \cdots \psi(l_n) \rangle = \lim_{\tau\to\infty} \langle 0| e^{-\tau H} \psi^+(l_1) \cdots \psi^+(l_n)|0\rangle$$

$$= \langle v| \psi^+(l_1) \cdots \psi^+(l_n)|0\rangle$$

$$\langle v| = \lim_{\tau\to\infty} \langle 0| e^{-\tau H} \quad \Rightarrow \quad \langle v|H = 0$$

$\leftarrow H|0\rangle = 0$

loop eq. $\langle v| [H, \psi^+(l_1) \cdots \psi^+(l_n)] |0\rangle = 0$

We have found

$$Z = \int d\phi' \int dm \; e^{-N \mathrm{tr} V(\phi') - 2\int_0^\infty \frac{dl}{l} e^{-lm} \psi(l)} .$$

We can express the dynamics of ϕ' by the Hamiltonian, and obtain

$$\langle \psi(l_1) \cdots \psi(l_n) \rangle$$

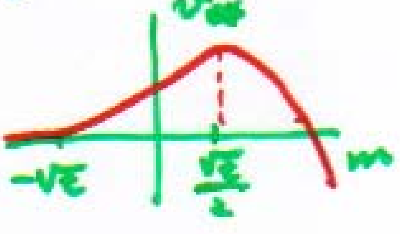
$$= \int dm \lim_{\tau\to\infty} \langle 0 | e^{-\tau H} \, e^{-2\int_0^\infty \frac{dl}{l} \psi^{+}(l) e^{-ml}} \times$$

$$\times \; \psi^{+}(l_1) \cdots \psi^{+}(l_n) \, | 0 \rangle$$

If we replace $\psi^{+}(l)$ on the exponential with the disk amplitude $\psi_0(l)$, we indeed obtain the following effective potential for m :

$$v_{eff}(m) = 2 \int \frac{dl}{l} \psi_0(l) e^{-ml} = -2 \int_{-\sqrt{t}}^{m} w_0(\zeta) .$$

$$w_0(\zeta) = (\zeta - \tfrac{1}{2}\sqrt{t}) \sqrt{\zeta + \sqrt{t}}$$



$m \simeq \frac{\sqrt{t}}{2}$ ⇒ one-instanton back ground

$\int dm$ ⇔ the original theory

continuum reformulation of $S^{(k)}$

$$\Phi = \left(\begin{array}{c|ccc} \phi' & & 0 & \\ \hline & m_1 & & \\ 0 & & \ddots & \\ & & & m_k \end{array}\right)$$

$$\underbrace{\qquad}_{N-k}\ \underbrace{\qquad\qquad}_{k}$$

$$\langle \varphi(l_1) \cdots \varphi(l_n) \rangle$$

$$= \int dm_1 \cdots dm_k \prod_{i<j} (m_i - m_j)^2 \cdot$$

$$\cdot \lim_{\tau\to\infty} \langle 0| e^{-\tau H} e^{-2\int_0^\infty \frac{dl}{l} \varphi^+(l) (e^{-m_1 l} + \cdots + e^{-m_k l})} \cdot$$

$$\cdot\, \varphi^+(l_1) \cdots \varphi^+(l_n) |0\rangle$$

$$= \int dm \lim_{\tau\to\infty} \langle 0| e^{-\tau H} e^{-2\int \frac{dl}{l} \varphi^+(l)\, \mathrm{tr}(e^{-ml})} \cdot$$

$$\cdot\, \varphi^+(l_1) \cdots \varphi^+(l_n) |0\rangle$$

m : $k \times k$ hermitian

In the planar limit,

$$e^{-2\int \frac{dl}{l} \varphi^+(l)\, \mathrm{tr}(e^{-ml})}$$

$$\to e^{-2\int \frac{dl}{l} \varphi_0(l)\, \mathrm{tr}(e^{-ml})} = e^{-\mathrm{tr}\, V_{eff}(m)} .$$

All of them are equivalent.

$k = 0$ $\Longleftrightarrow$ $k = \infty$

only closed string dual matrix model

another closed string matrix duality

5. Summary

- **The naive closed string field theory alone can not fix the normalization of the D-brane boundary state.**

$$\begin{aligned}\frac{\partial}{\partial\zeta}(w(\zeta))^2 &= \frac{1}{3}\zeta^2+\frac{4}{3}t\\ \rightarrow (w(\zeta))^2 &= (\zeta-\frac{1}{2}\sqrt{t})\sqrt{\zeta+\sqrt{t}}+\alpha\end{aligned}$$

D-brane coresponds to the integration constant, and can not be fixed by the loop equation, although the loop eq. is non-linear.

- **We can, however, give an equivalent formulation in which the instanton effects can be seen manifestly:**

$$\begin{aligned}&< w(l_1)\cdots w(l_n) >\\ &= \int dm \lim_{\tau\rightarrow\infty}\\ &\qquad < 0|e^{-\tau H}e^{-2\int\frac{dl}{l}\psi^\dagger(l)tr(e^{-ml})}\psi^\dagger(l_1)\cdots\psi^\dagger(l_n)|0 >\\ &\qquad\qquad m : k\times k \text{ matrix}\end{aligned}$$

Here all theories with different k are equivalent to the original theory, but the instanton effects can be easily seen in these formulations.

- **Furthermore, if we consider large k limit, the theory becomes a matrix model, which can be regarded as a "matrix model dual " of the original string theory.**

It would be interesting if one can derive the matrix model description of the critical string in a similar way.