Description of $\mathcal{N}=1$ Supersymmetric Gauge Theory from Whitham Integrable System and Gluino Condensate Prepotential

— dedicated to the memory of Bunji Sakita —

I) Introduction

- IK2 hep-th/0312*** with H. Kanno
- IM7 hep-th/0301136, IJMP with A. Morozov

cf.
IK1 hep-th/0304184, PLB
IM4 hep-th/0211245, NPB
IM5 hep-th/0211259, PLB
IM6 hep-th/0212032, PTP

- gauge theory with $\mathcal{N}=2$ susy has a successful exact description of LEEA in terms of

  the curve (RS),
  the meromorphic differential
  period integrals and
  prepotential

- shared by the corresponding classical integrable system of particles

- the case with $\mathcal{N}=1$ susy:
  $\boxed{}$ + fn $W(x=\Phi)$ that represents the superptl of the adjoint scalar Φ.

- $\exists$ a lot of cal on $\mathcal{N}=1$ gluino condensate using the matrix model & the attend. curve / differential

· Goal of today's talk:

is, however, **NOT** in computation/results.

Instead

is to understand this subject hopefully better

from **the PREPOTENTIAL VIEWPOINT**

with small theoretical discovery.

Contents:

II)

· **The curve** for $\mathcal{N}=2$, $SU(N)$ pure super-Yang Mills is a hyperelliptic Riemann surface of genus $N-1$:

$$Y^2 = P_N(x)^2 - 4\Lambda^{2N}$$

$$P_N(x) = \langle \det(x\mathbb{1} - \Phi) \rangle \equiv \prod_{i=1}^{N}(x-p_i) = x^N - \sum_{k=2}^{N} u_k x^{N-k}$$

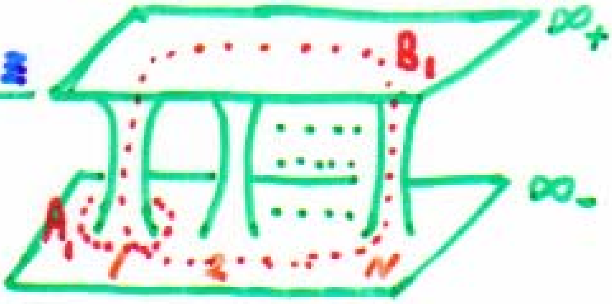
$\ln\det = \operatorname{tr}\ln \cdots\rightarrow$

$$= \sum_{k=0}^{N} \Delta_k(h_\ell)\, x^{N-k}$$

here $h_\ell = \frac{1}{\ell}\langle \operatorname{tr}_N \Phi^\ell \rangle = \frac{1}{\ell}\sum_{i=1}^{N} p_i^\ell$

$\ell = 2, \cdots N$ moduli

· Via the spectral parameter z the curve is written as



$$P_N(x) = z + \frac{\Lambda^{2N}}{z}$$

$$Y = z - \frac{\Lambda^{2N}}{z} \quad \Leftarrow \text{ } N \text{ site periodic Toda chain}$$

· **The distinguished meromorphic differential** for the prepotential theory is

$$d\hat{S}_{SW} = x\, d\log z = x\, t(x)\, dx, \quad t(x) = \frac{P_N'}{\sqrt{P_N^2 - 4\Lambda^{2N}}}$$

$\exists$ double pole at $\infty_\pm \leftrightarrow$ only T_1 turned on

· **The defining property** :

moduli derivatives are holomorphic

$$\frac{\partial}{\partial u_k} d\hat{S}_{SW}\Big|_{z,\Lambda} = \frac{x^{N-k}}{Y}\, dx$$

Q1 : Which one to fix?

or

$$\text{''}\ \Big|_{x,\Lambda} = \text{''} \quad -d\left(\frac{x^{N-k+1}}{Y}\right)$$

Q2 : What to regard as moduli ?

・(Generic) prepotential theory : Whitham deformations

deform both moduli of RS and the mero. differential consistently without losing the defining property :

$$d\hat{S}_{SW} \to d\hat{S}, \quad \frac{\partial}{\partial u_k} d\hat{S}\Big|_{*, \Lambda} = \text{holomorphic}$$

$\approx$

adding higher order poles to the original double pole

ξ : local coord. at these $\quad \xi = z^{\mp 1/N}$ or x^{-1}

Introduce $d\Omega_\ell$; a set of mero. diff

s.t. $d\Omega_\ell = \xi^{-\ell-1} d\xi$ + nonsingular $\quad \ell \geq 1$

In order to remove ambiguities, require

(*) $\oint_{A_i} d\Omega_\ell = 0$ $\quad$ cf. $d\omega_i$: canonical hol. diff. s.t. $\int_{A_i} d\omega_j = \delta_{ij}$

・ The upshot is

$$d\hat{S} = \sum_{i=1}^{g} a^i d\omega_i + \sum_{\ell \geq 1} T_\ell \, d\Omega_\ell$$

$\Rightarrow$ regard $h_k = h_k(a^i, T_\ell)$

$a^i = \oint_{A_i} d\hat{S}$; local coord. in the moduli space $\quad$ $T_\ell = \operatorname{res}_{\xi=0} \xi^\ell d\hat{S}$; time variables or T moduli

・ Introduce the prepotential $\mathcal{F}(a^i, T_\ell)$ via

(**) $\quad \dfrac{\partial \mathcal{F}}{\partial a^i} = \oint_{B^i} d\hat{S}, \quad \dfrac{\partial \mathcal{F}}{\partial T_\ell} = \dfrac{1}{2\pi i \ell} \operatorname{res} \xi^{-\ell} d\hat{S} \equiv \mathcal{H}_{\ell+1}(h_k)$

Picture we want to materealize as prepotential theory:

- $\mathcal{N}=2$ $\qquad\qquad$ $\mathcal{N}=1$

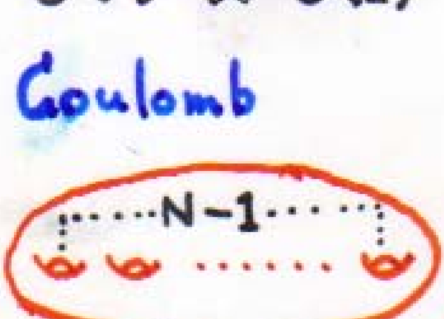
$$U(N) \text{ pure SYM} \xrightarrow{\qquad} \prod_{i=1}^{n} \times U(N_i)$$

deformed by superptl $g_{n+1}\int d^2\theta \, \mathrm{tr}\, W_{n+1}(\Phi)$ s.t. $W'_{n+1}(x) = \prod_{i=1}^{n}(x-\alpha_i)$

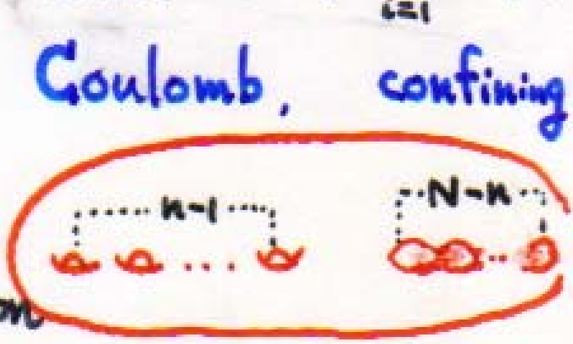
LEEA $\quad U(1)^{N-1} \times U(1) \longrightarrow U(1)^{n-1} \times U(1) \times \prod_{i=1}^{n} SU(N_i)$

Coulomb $\qquad\qquad$ Coulomb, confining

RS



$\longrightarrow$



degeneration
N−n non-intersecting cycles $\to 0$

$\mathcal{N}=1$ vacua is codimension $N-n$ subspace in $\mathcal{N}=2$ Coulomb branch and is parametrized by the order parameters (gluino condensates)

$$S_i \propto \mathrm{Tr}_{SU(N_i)} W^\alpha W_\alpha \qquad i=1\cdots n$$

III)

· Mixed second derivatives and the condition of det = 0

(**) $\Rightarrow$ $$\frac{\partial^2 \mathcal{F}}{\partial a^i \partial T_\ell} = \oint_{B^i} d\Omega_\ell = \frac{1}{2\pi i \ell} \operatorname{res} \zeta^{-\ell} d\omega_i \ , \quad i = 1 \sim N-1, \ \ell = 1 \sim N-1$$

We impose

(***) $$\det \frac{\partial^2 \mathcal{F}}{\partial a^i \partial T_\ell} = 0 .$$

· Implications.

- $\exists$ a nonvanishing column vector $\begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^{N-1} \end{pmatrix}$ s.t.

$$0 = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial T_\ell} c^\ell \underset{①}{=} \sum_{\ell=1}^{N-1} \oint_{B^i} d\Omega_\ell c^\ell \underset{②}{=} \frac{1}{2\pi i} \operatorname{res}_{\zeta=0} \left(\sum_{\ell=1}^{N-1} \frac{c^\ell}{\ell} \zeta^{-\ell} \right) d\omega_i$$

① $\Rightarrow$ $d\tilde{\Omega} \equiv \sum_{\ell=1}^{N-1} c^\ell d\Omega_\ell$ has vanishing periods over all A_i & B^i cycles.

$\therefore$ #(g) must decrease by the Riemann-Roch Th.

- $\exists$ a nonvanishing row vector $(\tilde{c}_1, \tilde{c}_2 \cdots \tilde{c}_{N-1})$ s.t. $$0 = \sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial^2 \mathcal{F}}{\partial a^i \partial T_\ell} = \sum \tilde{c}_i \frac{\partial \mathcal{H}_{\ell+1}}{\partial a^i}$$

$\therefore$ moduli depend actually on less than N-1 arguments

· Factorization/degeneration & the matrix model curve

$\therefore$ Let $n-1$ be $\#(g)$ after degeneration

$$\therefore \begin{cases} Y^2 = H_{N-n}(x)^2 F_{2n}(x) \\ P_N'(x) = H_{N-n}(x)\, R_{n-1}(x) \end{cases}$$

and $\; t(x) = R_{n-1}(x) \big/ \sqrt{F_{2n}(x)}$

Finally examine ②.

Let $\displaystyle\sum_{\ell=1}^{N-1} \frac{c_\ell}{\ell} z^{-\ell} \equiv \underbrace{W_{k+1}'(x)}_{\text{poly of degree } k\,(\geq n) \text{ in } x} \equiv \prod_{j=1}^{k}(x-\alpha_j)$

and $\dfrac{x^{j-1}}{\sqrt{F_{2n}}}$ serve as bases of holo. diff. of the reduced R.S.

$$\therefore \; 0 = \operatorname{res}_{x=\infty}\left(W_{k+1}'(x) \frac{x^{j-1}}{\sqrt{F_{2n}}} \right)$$

$$\therefore \; \frac{W_{k+1}'}{\sqrt{F_{2n}}} = \underbrace{Q_{k-n}}_{\text{deg.}}(x) + \sum_{\ell > n} \frac{\beta_\ell}{x^\ell}$$

$$\therefore \; y^2 \equiv F_{2n} Q_{k-n}^2 = W_{k+1}'^{\,2} + f_{k-1}$$

This is the curve appearing in the n-cut solution of the matrix model

IV)

· **Classical limit** $\Lambda = 0$ largely simplified

$$Y = z = \prod_{\ell=1}^{N} (x - p_\ell)$$

$$d\hat{S}_{SW}^{(class)} = x \sum_{i=1}^{N} \frac{1}{x-p_i} dx \qquad \therefore a_i^{(class)} = p_i$$

$$= \sum_{i=1}^{N} p_i \, d\omega_i + N dx \quad \left(\because d\omega_i^{(class)} = \frac{dx}{x-p_i} \right)$$

· **Suppose**

$$z = \prod_{j=1}^{n} (x - \beta_j)^{N_j} \; , \; \sum_{j=1}^{n} N_j = N$$

i.e. N_j poles coalesce at β_j $j = 1 \sim n$

$$d\omega_j^{(class, red)} = \frac{dx}{x - \beta_j}$$

the condition ②

$$0 = \operatorname*{res}_{x=\infty} \left(W'_{k+1}(x) \, d\omega_j^{(class. red)} \right) , \; j = 1 \sim n$$

(contour deformation)

β_j must coincide with one of the roots α_j of W'_{k+1}

The value of Φ constrained to the extremum of W_{k+1}

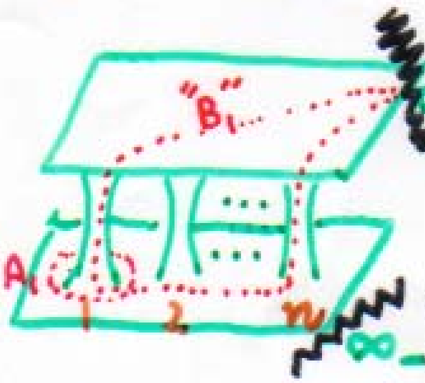
$\therefore W_{k+1}$ tree level superpotential

V)

We have the reduced curve of $g = n-1$

($k=n$ for simplicity)

$$y^2 = W'_{n+1}(x;\alpha_j)^2 + f_{n-1}(x)$$



denote the coeff by $b_\ell(\alpha_j)$

& temporarily forget the α dependence $\approx$ offshell

dim (moduli) = $2n$

$\approx$ cut lengths + cut positions

· What should $d\hat{S}_{mat}$ be?

IM4

demand **almost** holomorphic after b_ℓ derivatives are taken

cf bases $\dfrac{x^{j-1}}{y}$ $j=1,\cdots n-1, n$

Obviously

$$d\hat{S}_{mat} = y(x)\,dx \quad , \quad T_1, T_2, \cdots T_n\, T_{n+1} \text{ \& } g \text{ turned on}$$

period integral

$$S_i = \oint_{A_i} d\hat{S}_{mat} \quad i = 1\sim n$$

but no reason to set

$$S \equiv \sum_{i=1}^{n} S_i = \int_{\coprod_{i=1}^{n} \cup A_i} d\hat{S}_{mat}$$

equal to zero

(*) and

$$\frac{\partial \mathcal{F}}{\partial S_i} = 2\int_{i\ \text{edge}}^{\text{cutoff}} d\hat{S}_{mat}$$

$\Downarrow$

$\exists$ cutoff

IM6

Originally (*) + small cut expansion $\Longrightarrow$ answer for $\mathcal{F}$ to order S_i^3

· Calculus from T moduli

IM7

Yet, $\exists$ simpler procedure thanks to Whitham machinery

$$T_{m+1} = \mathrm{res}_\infty\, x^{-m-1}\, d\hat{S}_{mat} = g\, u_m \quad . \quad u_m = (-)^{n-m} e_{n-m}(\alpha)$$

$$e_m(\alpha) = \sum_{i_1<\cdots<i_m} \alpha_{i_1} \cdots \alpha_{i_m}$$

(★★) by experience

$$\frac{1}{g}\frac{\partial \mathcal{F}}{\partial u_\ell} = \frac{\partial \mathcal{F}}{\partial T_{\ell+1}} = \frac{1}{\ell+1}\mathrm{res}_\infty (x^{\ell+1} - \Lambda^{\ell+1})\, d\hat{S}_{mat}$$

and parameterize

$$f_{n-1}(x) = \sum_{i=1}^{n} \tilde{S}_i \prod_{j(\neq i)}^{n} (x-\alpha_j) = W'_{n+1}(x) \sum_{i=1}^{n} \frac{\tilde{S}_i}{x-\alpha_i}$$

- $d\hat{S}_{mat}$ expandable in $\tilde{S}_i$
- A_i cycle integrations & inversion give

$$\tilde{S}_i = S_i + \frac{1}{2g}\sum_{j,k} \frac{1}{\alpha_{ij}\alpha_{ik}\Delta_i} S_j S_k + \cdots$$

- $$\frac{\partial \alpha_j}{\partial u_m} = -\alpha_j^m / \Delta_j$$

$\Rightarrow$ R.H.S of (★★) $=$

$$\sum_i S_i \left(\frac{\partial W_{n+1}(\alpha_i)}{\partial u_\ell} - \frac{\partial W_{n+1}(\Lambda)}{\partial u_\ell} \right) - \frac{1}{4}\sum_{j<k} \left(S_j^2 + S_k^2 - 4 S_j S_k \right) \frac{\partial}{\partial u_\ell} \log \alpha_{jk} + \cdots$$

$\Rightarrow$ answer

· Proposed form of $\mathcal{F}(S|\alpha)$:

$$2\pi i\,\mathcal{F}(S|\alpha) = 4\pi i g_{n+1}\left(W_{n+1}(\Lambda)\sum_i S_i - \sum_i W_{n+1}(\alpha_i) S_i\right) - \left(\sum_i S_i\right)^2 \ln\Lambda$$
$$+\frac{1}{2}\sum_{i=1}^{n} S_i^2\left(\log\frac{S_i}{4}-\frac{3}{2}\right) - \frac{1}{2}\sum_{i<j}\left(S_i^2 - 4S_iS_j + S_j^2\right)\log\alpha_{ij} + \sum_{k=1}^{\infty}\frac{\mathcal{F}_{k+2}(S|\alpha)}{(i\pi g_{n+1})^k}$$

$k+2$ order poly. in S_i

· Our result:

IM6

$$\mathcal{F}_3(S|\alpha) = \sum_{i=1}^{n} u_i(\alpha) S_i^3 + \sum_{i(\neq j)} u_{i;j}(\alpha) S_i^2 S_j + \sum_{i<j<k} u_{ijk}(\alpha) S_i S_j S_k$$

$$u_i(\alpha) = \frac{1}{6}\left(-\sum_{j(\neq i)}\frac{1}{\alpha_{ij}^2\Delta_j} + \frac{1}{4\Delta_i}\sum_{\substack{j<k\\ j,k(\neq i)}}\frac{1}{\alpha_{ij}\alpha_{ik}}\right)$$

$$u_{i;j}(\alpha) = \frac{1}{4}\left(\frac{-3}{\alpha_{ij}^2\Delta_i} + \frac{2}{\alpha_{ij}^2\Delta_j} - \frac{2}{\alpha_{ij}\Delta_i}\sum_{k(\neq i,j)}\frac{1}{\alpha_{ik}}\right)$$

$$u_{ijk}(\alpha) = \frac{1}{\alpha_{ij}\alpha_{ik}\Delta_i} + \frac{1}{\alpha_{ji}\alpha_{jk}\Delta_j} + \frac{1}{\alpha_{ki}\alpha_{kj}\Delta_k}$$

$$\Delta_i \equiv W''_{n+1}(\alpha_i) = \prod_{j(\neq i)}^{n}\alpha_{ij}$$

originally done by the small cut expansion

VI)

Instead, let me recall what Antal Jevicki spoke at the banquet on October 1st, 1990.

" So I took a summer school course on group theory before joining CCNY.

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A few months later, I discovered that he does not know much on group theory .... much beyond the harmonic oscillator representation of generators.

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