Noncommutative Superspace and Supermatrix Models

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based on collaborations with H. Umetsu (KEK)
(partly with M. Hatsuda)

ref.

- M. Hatsuda, H. Umetsu S. I.
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In Memory of
Prof. Bunji Sakita

1 Introduction

⟨Stringy motivation⟩

NS $B_{\mu\nu}$ $\rightarrow$ $[x_\mu x_\nu] \neq 0$

RR bgd. $\rightarrow$ $\{\theta_\alpha \theta_\beta\} \neq 0$
(graviphoton)

Ooguri Vafa
Seiberg '03
de Boer et. al.

Hybrid formulation of superstring (Berkovits)

$$\mathcal{L} = \cdots + p_\alpha \partial\theta^\alpha + \tilde{p}_\alpha \partial\tilde{\theta}^\alpha + \alpha' F^{\alpha\beta} p_\alpha \tilde{p}_\beta + \cdots$$

RR field strength

$\Downarrow$

propagator on boundary

$$\langle \theta^\alpha(\tau)\, \theta^\beta(\tau') \rangle = (2\pi\alpha')^2 F^{\alpha\beta} \, \mathrm{sgn}(\tau - \tau'), \cdots$$

$\Downarrow$

NC superspace

$$\begin{cases} \{\theta^\alpha, \theta^\beta\} = i(2\pi\alpha')^2 F^{\alpha\beta} \\ [x^\mu x^\nu] = i 2\pi\alpha' \left(\dfrac{1}{g + 2\pi\alpha' B}\right)^{\mu\nu}_{AS} \end{cases}$$

Ooguri Vafa ('03)

⇨ Non-planar contributions to effective potential of N=1 SYM
(Gravitational F-term)

… David Gava Narain ('03)

Kawai Kuroki Morita

super matrix models → proof of DV conject.

(large N reduction)

<Historically>

- Noncommutativity of superspace

Kosinski et.al ('00), Ferrara et.al. ('00) …

- Realization of NC superspace by Supermatrix

Grosse et. al. ('95)

(Fuzzy supersphere S^{2} $osp(1/2)$)

scalar multiplet on super S^{2}

⇓

Our motivation is

to obtain NC superspace as a classical solution of supermatrix models and investigate small fluctuations around it.

(also Shibusa, Tada '03)

⟨Matrix Models and dynamical generation of Space⟩

- type IIB MM (IKKT '96)

$$S = tr([A_\mu A_\nu]^2 + \bar{\Psi}\gamma^\mu[A_\mu, \Psi])$$

$$A_\mu = \underset{\sim}{X_\mu} + \tilde{A}_\mu$$

classical solution = bgd. space-time

if $[X_\mu X_\nu] = i\,\theta_{\mu\nu} \Rightarrow \tilde{A}_\mu$: gauge th. on NC space

$[A_\mu A_\nu] = i(\theta_{\mu\nu} + \tilde{F}_{\mu\nu})$ Origin of SW map.

- How can we describe RR bgd. in matrix models ?

Can we obtain NC superspace as a class. sol. ?

⇩

Supermatrix Models

difference from NS $B_{\mu\nu}$

- Constant NS $B_{\mu\nu}$ ⇨ flat space is a sol. to string e.o.m.

- RR $F^{\alpha\beta}$

  except a special case ($F^{\alpha\beta} \neq 0$, $F^{\dot{\alpha}\dot{\beta}} = 0$)

  if ($F^{\alpha\beta} \neq 0$, $F^{\dot{\alpha}\dot{\beta}} \neq 0$) ⇨ Curved (E) $AdS^2 \times S^2$

⇩

In this talk, a simplest case

fuzzy supersphere S^2

NC

2. Superspace as a constrained system ($N = 1/2$)

· NC space $[\hat{x}, \hat{y}] = i\theta$ = Constrained system on lowest Landau level

charged particle in strong magnetic field in $d=2$

⇩

Lowest Landau level

$D_i = -i(\partial_i - ieA_i)$ $\qquad (D_x + iD_y)\psi_{LLL} = 0$

⇨ 2nd class constraints $D_i \approx 0$ $(i = 1, 2)$

$$[x, y]_D = [\hat{X}, \hat{Y}] = \frac{i}{B}$$

W_∞-algebra (Karabali Sakita s.l.) Also (Dhar Mandal Wadia) Shizuya Ezawa

· Covariant derivative on superspace

$$\begin{cases} D_\mu = \partial_\mu - \frac{i}{2} f_{\mu\nu} x^\nu \\ D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha \partial_\mu + \frac{i}{2} f_{\alpha\beta}\theta^\beta \\ \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu - \frac{i}{2} f_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}} \end{cases}$$

assume $f_{\mu\nu} = f_{\dot{\alpha}\dot{\beta}} = 0$ $\qquad f_{\alpha\beta} \neq 0$

$\{D_\alpha D_\beta\} = i f_{\alpha\beta}$

$$D_\alpha \approx 0 \Rightarrow \begin{cases} [x^\mu, x^\nu]_D = i f^{-1}_{\alpha\beta}(\sigma^\mu\bar{\theta})_\alpha(\sigma^\nu\bar{\theta})_\beta \\ [x^\mu \theta_\alpha]_D = -f^{-1}_{\alpha\beta}(\sigma^\mu\bar{\theta})_\beta \\ \{\theta_\alpha \theta_\beta\}_D = i f^{-1}_{\alpha\beta} \end{cases}$$

$N = \frac{1}{2}$ superspace

$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$

$\{\theta_\alpha \theta_\beta\}_D = i f^{-1}_{\alpha\beta}$, others = 0

3. Fuzzy supersphere S^{2}

Grosse et.al. ('95)

- $osp(1|2)$ $D=2,\ N=1$

$$[l_i\, l_j] = i\,\epsilon_{ijk}\, l_k \quad (i=1,2,3)$$
$$[l_i\, v_\alpha] = \left(\frac{\sigma_i}{2}\right)_{\beta\alpha} v_\beta \quad (\alpha=1,2)$$
$$\{v_\alpha\, v_\beta\} = \left(\frac{C\sigma_i}{2}\right)_{\alpha\beta} l_i \qquad C = i\sigma_2$$

- Irrep super spin $L \Rightarrow L \oplus (L-\frac{1}{2})$ for $SU(2)$

$$\dim = (2L+1) + 2L = 4L+1 \ (\equiv N)$$

$$l_i = \begin{pmatrix} l_i^{(L)} & \\ & l_i^{(L-\frac{1}{2})} \end{pmatrix} \quad v_\alpha = \begin{pmatrix} 0 & v_\alpha^{(L,L-\frac{1}{2})} \\ v_\alpha^{(L-\frac{1}{2},L)} & 0 \end{pmatrix}$$

- Coordinates $(x_i, \theta_\alpha) = (l_i, v_\alpha)$

$$(x_i)^2 + \theta^\alpha \theta_\alpha = L(L+\tfrac{1}{2}) = \text{const.}$$

- Fields on fuzzy supersphere

$\Downarrow$

Polynomials of (l_i, v_α) $\quad \Phi(l_i, v_\alpha)$

bases $Y^{S}_{km}(l_i, v_\alpha)$ $(k = 0, \frac{1}{2}, 1, \cdots 2L)$

$$\Phi(l,v) = \sum_{k=0,\frac{1}{2},\cdots}^{2L} \phi_{k,m}\, Y^{S}_{km}(l,v)$$

- $N=1$ $osp(1|2)$ can be extended to $osp(2|2)$ within $(N\times N)$ irrep. of $osp(1|2)$

$$\begin{cases} \gamma = -\dfrac{4}{4L+1}\left(C_{\alpha\beta} v_\alpha v_\beta + 2L(L+\tfrac{1}{2})\right) \\ d_\alpha = \dfrac{2}{4L+1} (\sigma_i)_{\beta\alpha} \{v_\beta\, l_i\} \end{cases}$$ covariant derivative

$$\left(\begin{array}{l} [\gamma, v_\alpha] = d_\alpha \quad [\gamma, d_\alpha] = v_\alpha \quad [\gamma, l_i] = 0 \\ [l_i, d_\alpha] = \left(\frac{\sigma_i}{2}\right)_{\beta\alpha} d_\beta ,\ \{d_\alpha\, d_\beta\} = -\left(\frac{C\sigma_i}{2}\right)_{\alpha\beta} l_i ,\ \{v_\alpha\, d_\beta\} = -\frac{C_{\alpha\beta}}{4}\gamma \end{array}\right)$$

$(C_{\alpha\beta} d_\alpha d_\beta + \gamma^2)$ commutes with $osp(1|2)$

- scalar superfield theory on fuzzy super S^2

$$\Phi = \phi + \psi_\alpha \theta^\alpha + (F + (x\partial)\phi)\theta^2$$

$$S = str\left(\Phi (C_{\alpha\beta} d_\alpha d_\beta + \gamma^2)\Phi + W(\Phi)\right)$$

4. Dynamical generation of NC super S^2

(Gauge theory on NC s.s^2)

4.1. Model (1)

$N = (2L+1) + 2L$

$M = A_i \otimes t_i + C_{\alpha\beta}\psi_\alpha \otimes q_\beta$ $\quad (A_i, \psi_\alpha)$ $N \times N$ matrix

(t_i, q_α) : 3×3 rep. of $osp(1|2)$

$$S = \underset{\substack{(3\times 3)\\(N\times N)}}{Str} (M^3 + \lambda M^2)$$

$$= \underset{(N\times N)}{Str} \left(\frac{i}{4}\epsilon_{ijk}A_iA_jA_k + \frac{\lambda}{2}A_iA_i - \frac{3}{16}\psi_\alpha(\sigma_i C)_{\alpha\beta}[A_i\,\psi_\beta] - \frac{\lambda}{2}C_{\alpha\beta}\psi_\alpha\psi_\beta \right)$$

$\begin{cases} \cdot\ osp(1|2) \text{ global susy} \\ \cdot\ \text{gauge sym. } U(2L+1|2L) \end{cases}$

$$\text{EOM} \begin{cases} i\epsilon_{ijk}A_jA_k + \frac{4}{3}\lambda A_i + \frac{1}{4}(\sigma_i C)_{\alpha\beta}\{\psi_\alpha\psi_\beta\} = 0 \\ \frac{3}{8}(\sigma_i C)_{\alpha\beta}[A_i\,\psi_\beta] + \lambda C_{\alpha\beta}\psi_\beta = 0 \end{cases}$$

$\Downarrow$

classical sol.

$$A_i^{cl} = \left(\frac{16}{9}\lambda\right) l_i \,, \quad \psi_\alpha^{cl} = \left(\frac{16}{9}\lambda\right) d_\alpha$$

$$A_i = \frac{16}{9}\lambda\,(\ell_i + \tilde{\hat{a}}_i) \qquad \psi_\alpha = \frac{16}{9}\lambda\,(d_\alpha + \tilde{\varphi}_\alpha)$$

$\Downarrow$

gauge th. on NC super S^2

$$\begin{cases} \tilde{a}_i = \underline{a_i} + \underline{\xi_{i\alpha}}\,\theta_\alpha + \underline{b_i}\,\theta^2 \\ \varphi_\alpha = \underline{\xi_\alpha} + (\sigma_\mu)_{\beta\alpha}\,\underline{c_\mu}\,\theta_\beta + \underline{\chi_\alpha}\,\theta^2 \end{cases}$$

- WZ like gauge $\quad \theta^\alpha \tilde{\varphi}_\alpha = 0$
- integrate out auxiliary fields (b_i, c_i, χ_α)

$\Downarrow$

$(a_i,\ \xi^{(3/2)}_{i\alpha})$

Supersym. gauge th. on S^2

4.2 Model (2)

$$M = \underline{A_i} \otimes t_i + C_{\alpha\beta} \underline{\Psi_\alpha} \otimes q_\beta - C_{\alpha\beta} \underline{\varphi_\alpha} \otimes d'_\beta - \frac{1}{4} \underline{W} \otimes \gamma'$$

$(t_i \; q_\alpha \; d'_\alpha \; \gamma')$ $osp(2|2)$ generators

generally · $G \Rightarrow (N=2)$ extension of H $\quad osp(2|2)$

· $H \Rightarrow (N=1)$ alg. $\quad osp(1|2)$

$$M = M_A \otimes T^A, \quad \{T^A\} = \mathfrak{g} \supset \{t^i\} = \mathfrak{h}$$

$$M = M_H + M_{H^\perp} \qquad M_H = M_i \otimes t^i$$

· product $\quad M \cdot N \equiv (-1)^{|T^A||N_B|} M_A N_B \otimes T^A T^B$

· $*$ product $\quad M * N \equiv (-1)^{|T^A||N_B|} M_A N_B \otimes [T^A T^B\}$

(not associative)

$$\mathrm{Str}(M \cdot N * L) = \mathrm{Str}(M * N \cdot L)$$

Field strength $\quad F \equiv M * M + \frac{d_G}{2} M \quad$ (d_G: Dynkin num.)

$$(f^{ABC} f_{AB}{}^{C'} = d_G \, 2^{CC'})$$

$$F_H \equiv M_H * M_H + \frac{d_H}{2} M_H$$

if M_A satisfy algebra $\mathfrak{g}$

$$F = F_H = 0$$

$$d_{osp(1|2)} = \frac{3}{2}, \quad d_{osp(2|2)} = 1$$

superfields $M_A = (\underbrace{A_i, \psi_\alpha}_{H}, \underbrace{\varphi_\alpha, w}_{H^\perp})$

$\Downarrow$

constraints

(1) $(F - F_H)|_H = 0$

$$\Rightarrow \begin{cases} A_i = (\sigma_i C)_{\alpha\beta} \{\varphi_\alpha\, \varphi_\beta\} \\ \psi_\alpha = [w, \varphi_\alpha] \end{cases}$$

(2) $\underset{(3\times 3)}{\mathrm{Str}}(M_{H^\perp} \cdot M_{H^\perp}) = 0$

W can be solved.

$\Downarrow$

independent variables φ_α

Action $S = (\mathrm{Str}(F \cdot F) + S_{CS}(M)) - (\mathrm{Str}(F_H \cdot F_H) + S_{CS}(M_H))$

$$S_{CS}(M) = \mathrm{Str}\left(\frac{2}{3} M \cdot M * M + \frac{d_G}{2} M \cdot M\right)$$

$$= \mathrm{Str}\left(M \cdot F - \frac{1}{3} M \cdot M * M \right)$$

$\downarrow$

classical sol. of EOM (satisfying constraints)

$$M_0 = l_i \otimes t_i + C_{\alpha\beta}\, v_\alpha \otimes q_\beta - C_{\alpha\beta}\, d_\alpha \otimes q_\beta' - \frac{1}{4} \gamma \otimes \gamma'$$

$$A_i = l_i + \tilde{a}_i,\ \varphi_\alpha = v_\alpha + \tilde{\varphi}_\alpha,\ \psi_\alpha = d_\alpha + \tilde{\psi}_\alpha,\ w = \gamma + \tilde{w}$$

$N = 2$ superspace connections

WZ gauge $\bar{\theta}^{\alpha} \tilde{\varphi}_{\alpha} = 0$

Gauge th. on $d=2$ fuzzy super sphere

$$\tilde{\varphi}_{\alpha} = (\sigma_i)_{\beta\alpha}\, \underline{a_i}\, \theta_{\beta} + \underline{\xi_{\alpha}}\, \theta^2$$

(a_i, ξ_{α}) vector multiplet

$(i=1,2,3)$

$\downarrow$

$$S' = \int d\Omega \Big((F_{ij})^2 + \xi_{\alpha} (\sigma_i C)_{\alpha\beta} L_i \xi_{\beta} + i \varepsilon_{ijk} a_i F_{jk} + \big((x \cdot a)^2 + C_{\alpha\beta} \xi_{\alpha} \xi_{\beta} \big) \Big)$$

$$(F_{ij} = L_i a_j - L_j a_i - i \varepsilon_{ijk} a_k)$$

5. Discussions

- NC superspace = constrained system

  physical meaning ?
  (superparticle in large RR bgd. ?)

  super W_∞-alg.

- Supermatrix Models

  $\begin{cases} \text{class. sol.} = \text{NC supersphere } S^{12} \\ \text{fluct. around it} \to \text{gauge th.} \end{cases}$

  (1) space = composite of fermions

  $$x_i = (C\sigma_i)_{\alpha\beta}\, \{\theta_\alpha\, \theta_\beta\}$$

  Nieuwenhuizen Schwarz ('82)

  (2) No flat limit with finite NC
  & ordinary susy alg.

· Higher dimensional Generalization

(1) $d = 4$ (flat)

covariant superspace approach

$(A_\alpha, A_{\dot\alpha}, A_\mu)$ superfields

⊕ constraints ① $F_{\alpha\dot\beta} = \{A_\alpha A_{\dot\beta}\} - \sigma_\mu^{\alpha\dot\beta} A_\mu = 0$

$A_\mu = \bar\sigma_\mu^{\dot\alpha\beta} \{A_{\dot\alpha} A_\beta\}$

② $F_{\alpha\beta} = \{A_\alpha A_\beta\} = 0$, $F_{\dot\alpha\dot\beta} = \{A_{\dot\alpha} A_{\dot\beta}\} = 0$

$A_\alpha = e^U D_\alpha e^{-U}$ $A_{\dot\alpha} = e^V \bar D_{\dot\alpha} e^{-V}$

⇩

gauge fix $U = 0$ $(A_\alpha = D_\alpha)$

(remaining gauge = chiral parameter)

Equivalence ?

· Can we relax the constraints ② ?

if $F_{\alpha\beta}$ = const. NC super space

(field strengths must satisfy Jacobi id.)

· $F_{\alpha\beta}$ ·· are not given by SYM condensation.

We need to introduce external fields other than SYM.

(RR field ?)

(2) $d=4$ curved

(E) $AdS_2 \times S^2$ with $(F_{\alpha\beta}, F_{\dot\alpha\dot\beta})$

$\downarrow$ sol. to strings

psu(2|2) supermatrix models

(3) $d=10$

A_α ⋯ superfields

on-shell (EOM) $\Leftrightarrow$ $F_{\alpha\beta} = \{A_\alpha A_\beta\} - \gamma_\mu^{\alpha\beta} A_\mu$

or

$\gamma^{\alpha\beta}_{\mu_1 \cdots \mu_5} \{A_\alpha A_\beta\} = 0$

$\Updownarrow$

$D = u^\alpha D_\alpha \qquad D^2 = 0$

$(u^\alpha \gamma_\mu^{\alpha\beta} u^\beta = 0)$

CS action for $d=10$ SYM

(Berkovits)

(4) direct derivation of supermatrix from superstring

IIB MM = matrix reg. of IIB schild action

$U(N) \leftarrow$ APD on WS

Supermatrix

$U(N|M) \leftarrow$ super APD on WS

superembedding ?