"Topological Strings and Nekrasov's formulas"

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§ Nekrasov's formula and TST

SU(2) gauge theory

$$Z_{inst} = \sum_{k=1} Z_k \Lambda^{4k}$$

$$Z_k = \sum_{\substack{\ell_{R_1}+\ell_{R_2} \\ =k}} \prod_{l,m=1,2} \prod_{i,j=1}^{\infty} \frac{\sinh\beta(a_{lm} + \hbar(\mu_{l,i} - \mu_{m,j} + j - i))}{\sinh\beta(a_{lm} + \hbar(j-i))}$$

sum over Young tables : random partitions

$\beta, \hbar, \Lambda$

$\beta \to 0$

$\hbar \to 0 \qquad Z = e^{\frac{1}{\hbar^2} F_0 + \cdots}$

$F_0 = F_{SW}$

agrees with Seiberg-Witten prepotential

We consider Nekrasov's formula with

$\beta \neq 0, \quad \hbar \neq 0$

Physical identification

$\beta = R$ radius of S^1 of 5-th dimension

5 dimensional gauge theory on $\mathbb{R}^4 \times S^1$

$q = e^{-2\beta\hbar}$

$= e^{i g_s}$ g_s: string coupling constant

$\hbar = 0$ coupling to gravity

① $\beta = 0, \ \hbar = 0$

4 dim. gauge theory $\quad$ IIA/CY_3

② $\beta \neq 0, \ \hbar = 0$

5 dim. gauge theory $\quad$ M/CY_3

counts genus$=0$ curves in CY_3

③ $\beta \neq 0,\ \hbar = 0$

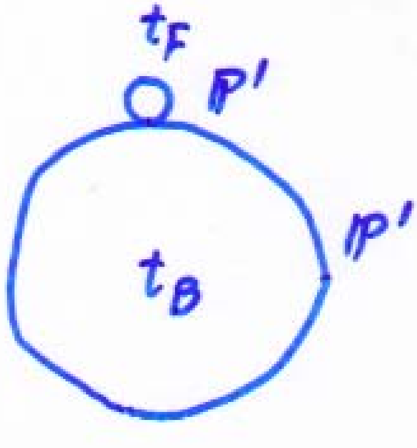
topological string on $K_{\mathbb{F}_0}$

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad K_{\mathbb{F}_0}$: canonical bundle on $\mathbb{F}_0$

$t_B,\ t_F$

$t_F = 4\beta a \to 0$

$Q_B = e^{-t_B} = (\beta\Lambda)^4, \quad t_B = \frac{1}{4}\log\frac{1}{\beta\Lambda} \to \infty$



A_1 singularity fibered over $\mathbb{P}^1$

We can compute $Z_{str}^{K_{\mathbb{F}_0}}$ using Chern-Simons theory.

We claim

$$Z_{str}^{K_{\mathbb{F}_0}} = Z_{Nek}^{SU(2)}$$

$$Z^{K_{\mathbb{F}_0}}_{str} = \exp\Big[\sum_{n,m} \sum_{g} \sum_{k} \frac{1}{k} \frac{N^g_{m,n}}{(\sin \frac{k g_s}{2})^{2-2g}} e^{-k(m t_B + n t_F)} \Big]$$

$N^g_{m,n}$: Gopakumar - Vafa invariants
counts genus g curves
winding m times base $\mathbb{P}^1$
n times fiber $\mathbb{P}^1$

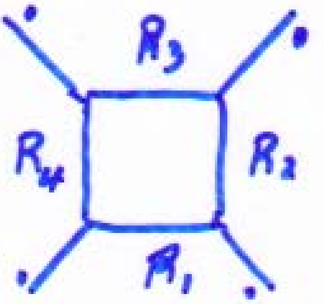
Nekrasov's formula encodes complete information on # of curves of all genus and all degrees.

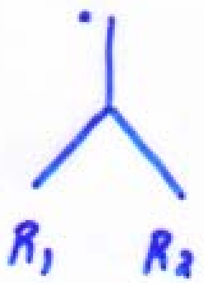
Similar formulas may be obtained also for local del Pezzo surfaces etc.

§ Chern-Simons calculation

local $\mathbb{F}_0$

toric diagram





$= W_{R_1 R_2}(q)$

$$Z_{str}^{K_{\mathbb{F}_0}} = \sum_{R_1,\cdots,R_4} W_{R_4 R_1} W_{R_1 R_2} W_{R_2 R_3} W_{R_3 R_4} \times e^{-t_F(l_{R_1}+l_{R_3})} e^{-t_B(l_{R_2}+l_{R_4})}$$

$W_{R_1 R_2}$: Hopf link inv. of CS Theory

$\langle \; {}^{R_1} \bigcirc\!\!\bigcirc {}^{R_2} \; \rangle_{CS}$ Wilson lines of reps. R_1 and R_2

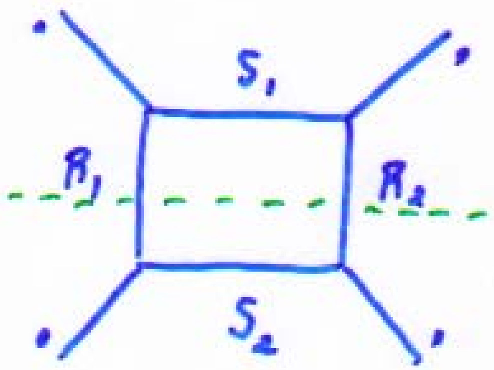
$q = e^{\frac{2\pi i}{N+k}}$ $\quad N \to \infty$, q : fixed

$$W_{R_1 R_2}(q) = W_{R_1}(q)\, S_{R_2}(x_i = q^{\mu_i^{R_1} - i + \frac{1}{2}})$$

Schur polynomial $\quad q^{\mu^{R_1}+\rho}$

$$W_R(q) = \dim_q R = S_R(q^\rho)$$

Method of computation



$$Q = e^{-t}$$

Define $\quad K_{R_1R_2}(Q) = \sum_S Q^{l_S} W_{R_1 S}(q) W_{S R_2}(q)$

$$Z_{str}^{K_{F_0}} = \cdots \sum_{R_1, R_2} (K_{R_1R_2}(Q_F))^2 e^{-t_B(l_{R_1}+l_{R_2})}$$

proposition 1

$$K_{R_1,R_2}(Q) = W_{R_1}(q) W_{R_2}(q) \exp\Big[\sum_{n=1} \frac{\tilde{f}_{R_1,R_2}(q^n)}{n} Q^n\Big]$$

$$\tilde{f}_{R_1,R_2} = \frac{(q-1)^2}{q} \bar{f}_{R_1}(q) \bar{f}_{R_2}(q)$$

$$\bar{f}_R(q) = f_R(q) + \frac{q}{(q-1)^2}$$

$$f_R(q) = \frac{q}{q-1} \sum_{i=0} (q^{\mu_i - i} - q^{-i})$$

$$\tilde{f}_{R_1R_2} = f_{R_1R_2} + \frac{q}{(q-1)^2}$$

$$f_{R_1R_2} = \frac{(q-1)^2}{q} f_{R_1} f_{R_2} + f_{R_1} + f_{R_2}$$

$$= \sum_k C_k(R_1,R_2) q^k$$

$$K_{R_1R_2}(Q_F) = W_{R_1}(q) W_{R_2}(q) \exp\left[\sum_{n=1} \frac{q^n}{(q^n-1)^2} \frac{Q_F^n}{n}\right]$$

$$\times \prod_k (1 - q^k Q_F)^{-C_k(R_1,R_2)}$$

$$\prod_k (1 - q^k Q_F)^{-C_k(R_1,R_2^t)}$$
$$= (4Q_F)^{-\frac{1}{2}(\ell_{R_1}+\ell_{R_2})} q^{-\frac{1}{4}(\kappa_{R_1}+\kappa_{R_2})} \prod_k \frac{1}{\sinh R(2a+\hbar k)^{C_k(R_1,R_2^t)}}$$

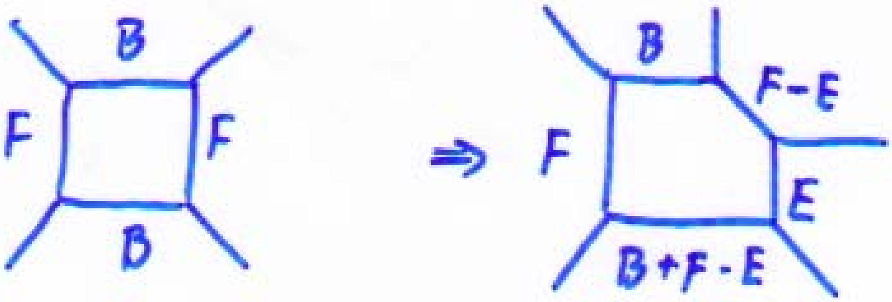
proposition 2

$$\prod_k \frac{1}{\sinh R(2a+\hbar k)^{C_k(R_1,R_2^t)}}$$

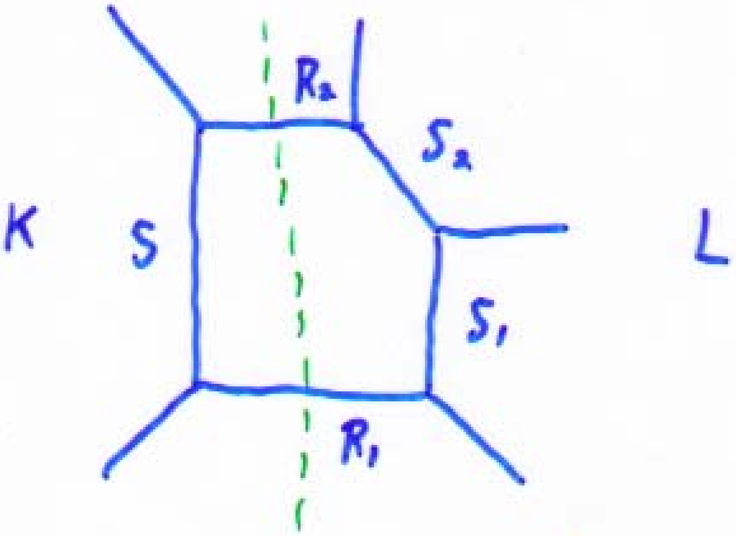
$$= \prod_{i,j=1}^{\infty} \frac{\sinh R(2a+\hbar(\mu_{1,i}-\mu_{2,j}+j-i))}{\sinh R(2a+\hbar(j-i))}$$

$$\therefore \quad Z_{str}^{K_{\mathbb{F}_0}} = Z_{Nek}^{SU(2)}$$

§ Adding matter



$\mathbb{F}_0$ $\qquad$ dP_2



$$K_{R_1 R_2}(Q) = \sum_S Q^{l_S} W_{R_1 S} W_{S R_2}$$

$$L_{R_1 R_2}(Q_1, Q_2) = \sum_{S_1, S_2} Q_1^{l_{S_1}} Q_2^{l_{S_2}} W_{R_1 S_1} W_{S_1 S_2} W_{S_2 R_2}$$

$$Z_{str}^{dP_2} = \sum_{R_1, R_2} Q_B^{l_{R_1} + l_{R_2}} K_{R_1 R_2}(Q_F) L_{R_1 R_2}(Q_F Q_E^{-1}, Q_E)$$

proposition 3

$$L_{R_1 R_2}(Q_1, Q_2)$$

$$= W_{R_1}(q) W_{R_2}(q) \exp\Big[-\sum_{n=1} \frac{\tilde{f}_{R_1}(q^n)}{n} Q_1^n - \sum_{n=1} \frac{\tilde{f}_{R_2}(q^n)}{n} Q_2^n + \sum_{n=1} \frac{\hat{f}_{R_1 R_2}(q^n)}{n} (Q_1 Q_2)^n \Big]$$

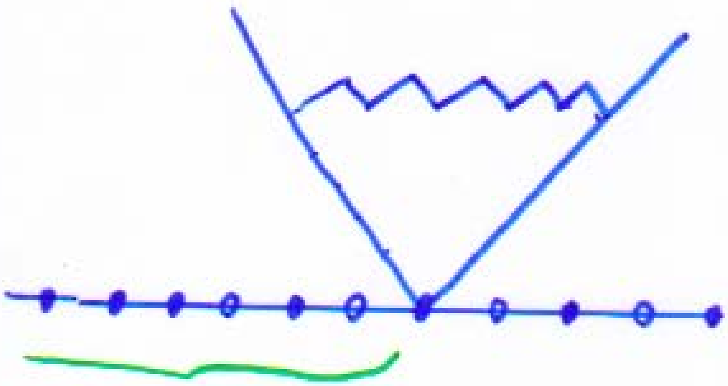
§ free fermions, vertex operators

proof of proposition 1, 3 :

Schur function calculus

We recall

Young diagrams $\Leftrightarrow$ free fermions



$\mu_1 = 5$, $\mu_2 = 4$

$\mu_3 = 3$, $\mu_4 = 2$

$\mu_5 = 1$

$\mu_i - i + \frac{1}{2}$: $\frac{9}{2}, \frac{5}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{7}{2}, -\frac{9}{2}, -\frac{11}{2}, \ldots$

$$|V_R\rangle = \psi^*_{\frac{9}{2}} \psi^*_{\frac{5}{2}} \psi_{-\frac{1}{2}} \psi_{-\frac{5}{2}} \underset{\overset{|||}{|0\rangle}}{|\,\mathrm{sea}\rangle}$$

$$S_R(x_i) = \langle V_R | \exp \sum_{n=1} \frac{p_n \alpha_{-n}}{n} | 0 \rangle$$

$$p_n \equiv \sum_{i=1} x_i^n$$

skew Schur function

$$S_{R/Q}(x_i) = \langle V_R | \, e^{\sum_{n \geq 1} \frac{p_n \alpha_{-n}}{n}} \, | V_Q \rangle$$

$$R \supset Q$$

Schur function calculus

= vertex operator algebra

sum over partitions

= sum over fermion Fock space

chern - Simons calculation

$\Rightarrow$ CFT correlation functions

We want to prove

GV invariants

study

more general geometric Transitions